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GAP AND DENSITY THEOREMS

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Dedicated to

NORBERT WIENER

by one of the many men whose careers he launched

PREFACE

The present volume is in many respects a companion volume to the earlier Colloquium Publication of Paley and Wiener. The same tools are used in both. (See Appendix.) Certain topics such as the closure of trigonometric sequences and some problems of Pólya are treated in both. The technique of the Fourier transform in the complex domain developed in Paley and Wiener has, in the present volume, been further extended in the solution of such problems as the general unrestricted gap Tauberian theorem.

The contents break up into four parts. Large portions of the first two parts have appeared in various journals during the past four years. The last two parts consist almost entirely of new material. The first part consists of Chapters I, II, III, and IV; the second part of Chapters V, VI, VII; the third part of Chapters VIII, and IX; and the fourth part of Chapters X, XI, and XII.

The present volume covers the various topics considered in some detail, most results being "best possible." Nevertheless for many of the topics treated there are several directions for further work in refining or developing these topics.

I am very much indebted to Professor J. D. Tamarkin for the painstaking manner in which he has examined and criticized this book.

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CHAPTER I

ON THE CLOSURE OF $\{e^{i\lambda_n x}\}$, I

1. Introduction. Among the basic facts known about the sequence of functions $\{e^{inx}\}$, $-\infty < n < \infty$, are

1. $\{e^{inx}\}$, $-\infty < n < \infty$, is closed over $(-\pi, \pi)$; that is, if $f(x) \in L(-\pi, \pi)$ then

$$\int_{-\pi}^{\pi} f(x)e^{inx}\,dx = 0, \qquad -\infty < n < \infty,$$

implies that $f(x)$ is a null function.

2. To a function $f(x) \in L(-\pi, \pi)$ there belongs the Fourier series

$$\sum_{-\infty}^{\infty} a_n e^{inx}, \qquad a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

This series will under certain conditions converge to the function $f(x)$, and under other conditions, although it may diverge, can nevertheless be summed to the function $f(x)$.

In the first four chapters we shall generalize these properties of $\{e^{inx}\}$ to sequences $\{e^{i\lambda_n x}\}$ with various conditions on $\{\lambda_n\}$. As is often the case these general results will, among other things, lead us back to certain properties of expansions in $\{e^{inx}\}$.

The closure of $\{e^{i\lambda_n x}\}$ can be related at once to theorems on analytic functions. For suppose that a sequence $\{e^{i\lambda_n x}\}$ is not closed. This means that there is at least one function, $f(x)$, not equivalent to zero such that for all λ_n of the sequence

$$\int_{-\pi}^{\pi} f(x)e^{i\lambda_n x}\,dx = 0.$$

Clearly

$$F(w) = \int_{-\pi}^{\pi} f(x)e^{iwx}\,dx$$

is an entire function and $F(\lambda_n) = 0$. Thus the study of closure amounts to the study of the zeros of certain entire functions.[1]

Theorems on closure lead at once to gap theorems.[2] For suppose a function

[1] This fact was first pointed out by Szász, Mathematische Annalen, vol. 77 (1916), p. 482.

[2] This was first pointed out by Paley and Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, vol. 19, 1934, Theorem XXXVI.

$f(x)$ which vanishes over $(\pi - L, \pi)$, $L > 0$, is expanded into a Fourier series, that is,

$$f(x) \sim \sum_{-\infty}^{\infty} a_n e^{inx}.$$

A gap condition asserts that certain a_n are zero; that is, $a_{\lambda_n} = 0$ for a sequence of integers $\{\lambda_n\}$. From the definition of a_n this implies that

$$\int_{-\pi}^{\pi-L} f(x) e^{-i\lambda_n x}\, dx = 0.$$

Thus we are back to the question of closure. In this way every theorem on closure will have as a corollary a gap theorem which will assert that if certain a_n in the Fourier series of $f(x)$ are zero then $f(x)$ cannot vanish over an interval of length L (where L depends on the sequence of integers for which a_n vanishes) unless $f(x)$ vanishes identically.

2. A closure theorem for $\lambda_n > 0$. The sequence of functions $\{e^{inx}\}$ is closed over $(-\pi, \pi)$ but is not closed over an interval of length $2\pi + a$ where $a > 0$. For let

$$f(x) = \begin{cases} -1, & -(\pi + \tfrac{1}{2}a) < x < -(\pi - \tfrac{1}{2}a), \\ 0, & -(\pi - \tfrac{1}{2}a) < x < (\pi - \tfrac{1}{2}a), \\ 1, & (\pi - \tfrac{1}{2}a) < x < (\pi + \tfrac{1}{2}a). \end{cases}$$

Then $f(x)$ is not equivalent to zero. Let

$$F(w) = \int_{-\pi-a/2}^{\pi+a/2} f(x) e^{iwx}\, dx.$$

A simple calculation gives

$$F(w) = \frac{4i}{w} \sin \pi w \sin \tfrac{1}{2}aw.$$

Since $\sin \pi n = 0$, it follows that $F(n) = 0$, $(-\infty < n < \infty)$. That is

$$\int_{-\pi-a/2}^{\pi+a/2} f(x) e^{inx}\, dx = 0, \qquad -\infty < n < \infty.$$

Since $f(x)$ is not equivalent to zero, it follows that $\{e^{inx}\}$ is not closed over an interval exceeding 2π in length. On the other hand if any term of $\{e^{inx}\}$, $(-\infty < n < \infty)$, is removed, the sequence ceases to be closed over the interval 2π.

We see therefore that $\{e^{inx}\}$, $(-\infty < n < \infty)$, is closed over an interval of length 2π and no greater one and that it ceases to be closed over $(-\pi, \pi)$ if any term is removed. Suppose we now delete about one-half of the sequence $\{e^{inx}\}$, $(-\infty < n < \infty)$, by considering the terms with positive n only. Then it seems likely that this sequence is closed over half as large an interval as

when n takes both positive and negative values, that is, over an interval of length π, and not over a longer one. However this is not the case. Actually the sequence $\{e^{inx}\}$, $(n > 0)$, is closed over any interval of length less than 2π. This follows from the following results.

THEOREM I. *Let $\Lambda(u)$ be the number of $\lambda_n < u$. Then the sequence $\{e^{i\lambda_n x}\}$, $(\lambda_n > 0)$ is closed over an interval of length L if*

$$\limsup_{R\to\infty} \left\{ \int_1^R \frac{\Lambda(u)}{u^{1/2}} \left(\frac{1}{u} + \frac{1}{R} \right) du - \frac{4L}{3\pi} R^{1/2} \right\} = \infty. \tag{2.01}$$

A corollary of Theorem I is

THEOREM II. *The sequence $\{e^{i\lambda_n x}\}$, $(\lambda_n > 0)$, is closed over an interval of length L if*

$$\liminf_{n\to\infty} \frac{n}{\lambda_n} > \frac{L}{2\pi}. \tag{2.02}$$

In case $\lambda_n = n$, it follows at once from (2.02) that $\{e^{inx}\}$, $(n > 0)$, is closed over any interval of length less than 2π.

Theorem I is, in case $\lim n/\lambda_n$, $(n \to \infty)$, exists, the best test to make for the closure of a sequence with $\lambda_n > 0$. For example, it follows easily from (2.01) that $\{e^{i\lambda_n x}\}$ with

$$\lambda_n = n - n^{1/2}, \qquad n > 1, \tag{2.03}$$

is closed over the interval 2π.

Theorem II[3] is much weaker than Theorem I. Later it will be replaced by far sharper results. We shall show that limit inferior in (2.02) can be replaced by limit superior (and even more) and the theorem remains true.

Proof of Theorem I. Let us assume that Theorem I is not true. Then (2.01) is satisfied but $\{e^{i\lambda_n x}\}$, $(\lambda_n > 0)$, is not closed over $(-\frac{1}{2}L, \frac{1}{2}L)$. Thus there exists a function $f(x)$, not equivalent to zero, such that

$$\int_{-L/2}^{L/2} f(x) e^{i\lambda_n x}\, dx = 0. \tag{2.04}$$

Let

$$F(w) = \int_{-L/2}^{L/2} f(x) e^{iw^2 x}\, dx.$$

Since $\{e^{2\pi inx/L}\}$, $(-\infty < n < \infty)$, is closed over $(-\frac{1}{2}L, \frac{1}{2}L)$, $F(w)$ cannot vanish at all points $w^2 = 2\pi n/L$, $(-\infty < n < \infty)$, for this would imply that

[3] In 1931 Pólya set as a problem the proof of Theorem II with $\{e^{\pm i\lambda_n x}\}$, that is, with positive as well as negative λ_n. Pólya, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 40 (1931), Problem 108. Many solutions have been given.

$f(x)$ is zero almost everywhere. Thus $F(w)$ is an entire function not identically zero. By (2.04), $F(\lambda_n^{1/2}) = 0$ for all λ_n of the sequence.

If $w = re^{i\theta}$, then

$$|F(re^{i\theta})| \leqq \int_{-L/2}^{L/2} |f(x)| e^{r^2|x \sin 2\theta|} dx$$

or

$$|F(re^{i\theta})| \leqq e^{(1/2)Lr^2|\sin 2\theta|} \int_{-L/2}^{L/2} |f(x)|\, dx. \tag{2.05}$$

Let $w_n = r_n e^{i\theta_n}$ denote the zeros of $F(w)$ in the right half-plane. Applying Carleman's theorem, Theorem B,[4] to $F(w)$, we have, if $w = u + iv$,

$$\sum_{r_n<R} \left(\frac{1}{r_n} - \frac{r_n}{R^2}\right) \cos\theta_n < \frac{1}{2\pi} \int_1^R \left(\frac{1}{v^2} - \frac{1}{R^2}\right) \log |F(iv)F(-iv)|\, dv$$
$$+ \frac{1}{\pi R} \int_{-\pi/2}^{\pi/2} \log |F(Re^{i\theta})| \cos\theta\, d\theta + A$$

where, in the proof of this theorem, we shall use A to represent any constant that depends only on $f(x)$. Using (2.05), we have

$$\sum_{r_n<R} \left(\frac{1}{r_n} - \frac{r_n}{R^2}\right) \cos\theta_n < \frac{A}{2\pi} \int_1^R \left(\frac{1}{v^2} - \frac{1}{R^2}\right) dv$$
$$+ \frac{1}{\pi R} \int_{-\pi/2}^{\pi/2} \tfrac{1}{2}LR^2 |\sin 2\theta| \cos\theta\, d\theta + A$$
$$< \frac{LR}{\pi} \int_0^{\pi/2} \sin 2\theta \cos\theta\, d\theta + A$$
$$< \frac{2LR}{3\pi} + A.$$

Since $w = \lambda_n^{1/2}$ are zeros of $F(w)$ in the right half-plane and therefore form a part of all the zeros, $\{r_n e^{i\theta_n}\}$, it follows that

$$\sum_{\lambda_n^{1/2}<R} \left(\frac{1}{\lambda_n^{1/2}} - \frac{\lambda_n^{1/2}}{R^2}\right) < \frac{2LR}{3\pi} + A.$$

If we replace R by $R^{1/2}$ in the above inequality, it becomes

$$\sum_{\lambda_n<R} \left(\frac{1}{\lambda_n^{1/2}} - \frac{\lambda_n^{1/2}}{R}\right) < \frac{2LR^{1/2}}{3\pi} + A. \tag{2.06}$$

If $\Lambda(u)$ is the number of $\lambda_n < u$, then (2.06) can be written as

$$\int_0^R \left(\frac{1}{u^{1/2}} - \frac{u^{1/2}}{R}\right) d\Lambda(u) < \frac{2LR^{1/2}}{3\pi} + A.$$

[4] See Appendix.

Integrating by parts, we obtain

$$\int_1^R \frac{\Lambda(u)}{u^{1/2}}\left(\frac{1}{u}+\frac{1}{R}\right)du - \frac{4LR^{1/2}}{3\pi} < A.$$

But this contradicts (2.01). Thus Theorem I is proved.

Theorem II is a consequence of Theorem I.

Proof of Theorem II. Since

$$\liminf n/\lambda_n > L/2\pi, \tag{2.02}$$

it follows that for large n and for some $\epsilon > 0$,

$$\left(\frac{L}{2\pi}+\epsilon\right)\lambda_n < n$$

or for large u, $u > R_0$,

$$\Lambda(u) > u\left(\frac{L}{2\pi}+\epsilon\right), \qquad \epsilon > 0.$$

Thus

$$\int_1^R \frac{\Lambda(u)}{u^{1/2}}\left(\frac{1}{u}+\frac{1}{R}\right)du > \int_{R_0}^R \left(\frac{L}{2\pi}+\epsilon\right)u^{1/2}\left(\frac{1}{u}+\frac{1}{R}\right)du$$
$$= \left(\frac{L}{2\pi}+\epsilon\right)\left(\frac{8R^{1/2}}{3} - 2R_0^{1/2} - \frac{2}{3}\frac{R_0^{3/2}}{R}\right).$$

Or

$$\int_1^R \frac{\Lambda(u)}{u^{1/2}}\left(\frac{1}{u}+\frac{1}{R}\right)du - \frac{4L}{3\pi}R^{1/2} \geqq \left(\frac{L}{2\pi}+\epsilon\right)\frac{8R^{1/2}}{3} - \frac{4L}{3\pi}R^{1/2} - 10LR_0^{1/2}$$
$$\geqq \frac{8\epsilon}{3}R^{1/2} - 10LR_0^{1/2}.$$

Thus as $R \to \infty$ (2.01) is satisfied and therefore (2.02) implies (2.01). This completes the proof of Theorem II.

3. A closure theorem for $-\infty < \lambda_n < \infty$. In this section[5] we shall investigate the closure of $\{e^{i\lambda_n x}\}$, $(-\infty < \lambda_n < \infty)$, that is, with λ_n taking on negative as well as positive values. We shall insist that our closure theorems shall include the well known fact that $\{e^{inx}\}$, $(-\infty < n < \infty)$, is closed over $(-\pi, \pi)$. Since $\{e^{inx}\}$, $(n > 0)$, is not closed over $(-\pi, \pi)$, we shall get a two-sided theorem (λ_n taking on negative as well as positive values) that is not true for positive values of λ_n only.

Certain refinements are necessary in the statement of the two-sided theorem which were not necessary in Theorem I. In part the reason for this is that the one-sided condition, (2.01), is not affected by the removal of any finite number of λ_n whereas in our simplest example of the two-sided cases, $\{e^{inx}\}$,

[5] Cf. Levinson, *On the closure of* $\{e^{i\lambda_n x}\}$, Duke Mathematical Journal, vol. 2 (1936), p. 51.

$(-\infty < n < \infty)$, deleting a single term affects the closure. One of the necessary refinements is that of closure L^p.

A set $\{e^{i\lambda_n x}\}$ is said to be closed $L^p(-\pi, \pi)$ if, for any $f(x) \in L^p(-\pi, \pi)$,

$$\int_{-\pi}^{\pi} f(x) e^{i\lambda_n x}\, dx = 0$$

for all λ_n implies that $f(x)$ is zero almost everywhere.

The basic and sharpest criterion is given by

THEOREM III. *Let $\Lambda(u)$ be the number of $|\lambda_n| \leqq u$. If*

$$\int_1^v \frac{\Lambda(u)}{u}\, du > 2v - \frac{p-1}{p} \log v - C \tag{3.01}$$

for some constant C, then the set $\{e^{i\lambda_n x}\}$, $(-\infty < n < \infty)$, is closed $L^p(-\pi, \pi)$, $p \geqq 1$.

A corollary of Theorem III is

THEOREM IV. *If*

$$|\lambda_n| \leqq |n| + \tfrac{1}{2}N + \frac{p-1}{2p}, \qquad -\infty < n < \infty, \tag{3.02}$$

the[6] *set $\{e^{i\lambda_n x}\}$, if it is not closed, becomes closed in adjoining to it at most any N terms $e^{i a_n x}$, $(1 \leqq n \leqq N)$. In particular, then, a sufficient condition for closure $L^p(-\pi, \pi)$, $p \geqq 1$, is*

$$|\lambda_n| \leqq |n| + \frac{p-1}{2p}, \qquad -\infty < n < \infty. \tag{3.03}$$

It follows from (3.03) that the sequence $\{e^{inx}\}$, $(-\infty < n < \infty)$, is closed $L(-\pi, \pi)$. [Condition (3.01) gives the even sharper result that $\{e^{i\lambda_n x}\}$, $(-\infty < n < \infty)$, is closed $L(-\pi, \pi)$ if $|\lambda_n| \leqq |n| + (\log |n|)^{-1-\delta}$, $|n| \to \infty$, for some $\delta > 0$.]

That Theorem IV is best possible follows from Theorem V.

THEOREM V. *If* (3.02) *is replaced by*

$$|\lambda_n| < |n| + \tfrac{1}{2}N + \frac{p-1}{2p} + \delta, \tag{3.04}$$

where $\delta > 0$, there exist sets $\{e^{i\lambda_n x}\}$ satisfying (3.04) *which do not become closed when N terms are adjoined to them. Thus the inequality* (3.02) *is a best possible result.*

In connection with these theorems the following result is of interest. It holds with no restrictions on the $\{\lambda_n\}$.

[6] In the case of closure in $L^2(-\pi, \pi)$, Paley and Wiener (loc. cit., chap. 6) proved that the set $\{1, e^{\pm i\lambda_n x}\}$ becomes closed on adjoining at most N terms if $|\lambda_n - n| < \frac{1}{4}N + \frac{1}{4}$.

THEOREM VI. *If the set $\{e^{i\lambda_n x}\}$ is closed $L^p(-\pi, \pi)$, $p \geqq 1$, it remains closed if we replace any λ_n by some other number.*

We shall now prove these theorems.

Proof of Theorem III. Let us suppose the theorem is not true. Then there exists an $f(x) \in L^p(-\pi, \pi)$ and not equivalent to zero such that

$$H(w) = \int_{-\pi}^{\pi} f(x) e^{iwx}\, dx \tag{3.05}$$

vanishes for $w = \lambda_n$, $(-\infty < n < \infty)$. Since $f(x)$ is not equivalent to zero, $H(w)$ is not identically zero.

Denote by $n(r)$ the number of zeros of $H(w)$ not exceeding r in magnitude. Then by Jensen's theorem, Theorem A,

$$\int_1^r \frac{n(u)}{u}\, du \leqq \frac{1}{2\pi} \int_0^{2\pi} \log |H(re^{i\theta})|\, d\theta + A, \tag{3.06}$$

where A will be used throughout the remainder of this section to represent various constants depending only on $f(x)$ and $\{\lambda_n\}$. By (3.05)

$$H(re^{i\theta}) = O(e^{\pi r |\sin \theta|}).$$

Using this in (3.06), we have

$$\int_1^r \frac{n(u)}{u}\, du \leqq 2r + A. \tag{3.07}$$

From this it follows, since $n(u)$ is an increasing function of u, that

$$n(r) \int_r^{2r} \frac{du}{u} \leqq \int_r^{2r} \frac{n(u)}{u}\, du \leqq \int_1^{2r} \frac{n(u)}{u}\, du < 4r + A.$$

Thus $n(r) < 4r/\log 2 + A$. Since $\{\lambda_n\}$ form part of the zeros of $H(w)$, $\Lambda(r) \leqq n(r)$. Thus

$$\Lambda(r) < \frac{4}{\log 2} r + A. \tag{3.08}$$

Let us assume that no λ_n is zero. (How to proceed if one of them is zero will be obvious.) Let

$$F(w) = \prod_{-\infty}^{\infty} \left(1 - \frac{w}{\lambda_n}\right) e^{w/\lambda_n}. \tag{3.09}$$

That $F(w)$ exists follows from (3.08). Since $H(w)$ vanishes at λ_n,

$$\phi(w) = H(w)/F(w)$$

is an entire function. If $n_1(r)$ is the number of zeros of $\phi(w)$ not exceeding r in magnitude, then clearly

$$n_1(r) = n(r) - \Lambda(r).$$

Thus

$$\int_1^r \frac{n_1(u)}{u}\,du = \int_1^r \frac{n(u)}{u}\,du - \int_1^r \frac{\Lambda(u)}{u}\,du.$$

Using (3.01) and (3.07), it follows that

$$\int_1^r \frac{n_1(u)}{u}\,du \leqq \frac{p-1}{p}\log r + A.$$

Since $(p-1)/p < 1$ it is clear that $n_1(r) = 0$, or in other words that $\phi(w)$ had no zeros. By the Hadamard factorization theorem, Theorem D, it follows that $H(w) = ae^{bu}F(w)$ and, in particular, that

$$|H(iv)| = e^{cv}\,|aF(iv)| \tag{3.10}$$

where a, b, and c are constants.

From (3.05), for any small $\epsilon > 0$,

$$|H(iv)| \leqq \left(\int_{-\pi}^{-\pi+\epsilon} + \int_{-\pi+\epsilon}^{\pi-\epsilon} + \int_{\pi-\epsilon}^{\pi}\right) e^{-vx}\,|f(x)|\,dx. \tag{3.11}$$

Using Holder's inequality, if $p > 1$,

$$|H(iv)| \leqq \left[\int_{-\pi+\epsilon}^{\pi-\epsilon} e^{-vxp/(p-1)}\,dx\right]^{(p-1)/p}\left[\int_{-\pi+\epsilon}^{\pi-\epsilon} |f(x)|^p\,dx\right]^{1/p}$$
$$+\left[\int_{-\pi}^{-\pi+\epsilon} e^{-vxp/(p-1)}\,dx\right]^{(p-1)/p}\left[\int_{-\pi}^{-\pi+\epsilon} |f(x)|^p\,dx\right]^{1/p}$$
$$+\left[\int_{\pi-\epsilon}^{\pi} e^{-vxp/(p-1)}\,dx\right]^{(p-1)/p}\left[\int_{\pi-\epsilon}^{\pi} |f(x)|^p\,dx\right]^{1/p}.$$

Or

$$|H(iv)| \leqq e^{(\pi-\epsilon)|v|}\,|v|^{-(p-1)/p}\,2\left[\int_{-\pi+\epsilon}^{\pi-\epsilon} |f(x)|^p\,dx\right]^{1/p}$$
$$+ e^{\pi|v|}\,|v|^{-(p-1)/p}\,4\left[\left(\int_{-\pi}^{-\pi+\epsilon} + \int_{\pi-\epsilon}^{\pi}\right)|f(x)|^p\,dx\right]^{1/p}.$$

For any δ we can choose an ϵ so that

$$\left[\left(\int_{-\pi}^{-\pi+\epsilon} + \int_{\pi-\epsilon}^{\pi}\right)|f(x)|^p\,dx\right]^{1/p} \leqq \delta.$$

Thus

$$|H(iv)| \leqq Ae^{\pi|v|}\,|v|^{-(p-1)/p}(e^{-\epsilon|v|} + \delta). \tag{3.12}$$

In case $p = 1$, (3.12) follows directly from (3.11). Thus (3.12) holds for all $p \geqq 1$. From (3.12)

$$\log |H(iv)| \leqq \pi |v| - \frac{p-1}{p} \log |v| + \log (e^{-\epsilon|v|} + \delta) + A.$$

Since δ can be chosen arbitrarily small, it follows that

$$\lim_{|v|\to\infty} \left(\log |H(iv)| - \pi |v| + \frac{p-1}{p} \log |v| \right) = -\infty.$$

Using (3.10), this gives

$$\lim_{|v|\to\infty} \left(\log |F(iv)| + cv - \pi |v| + \frac{p-1}{p} \log |v| \right) = -\infty. \tag{3.13}$$

On the other hand from the definition of $F(w)$, (3.09),

$$\log |F(iv)| = \frac{1}{2} \int_0^\infty \log (1 + v^2/u^2)\, d\Lambda(u).$$

Integrating by parts twice gives

$$\log |F(iv)| = \int_0^\infty \frac{2v^2 u}{(u^2 + v^2)^2}\, du \int_0^u \frac{\Lambda(y)}{y}\, dy.$$

Using (3.01), this becomes

$$\log |F(iv)| \geqq \int_0^\infty \frac{2v^2 u}{(u^2 + v^2)^2} \left(2u - \frac{p-1}{p} \log u - A \right) du.$$

Setting $u = |v| y$ gives

$$\begin{aligned} \log |F(iv)| &\geqq \pi |v| - \frac{p-1}{p} \log |v| \int_0^\infty \frac{2y}{(1+y^2)^2}\, dy - A \\ &\geqq \pi |v| - \frac{p-1}{p} \log |v| - A. \end{aligned}$$

But this contradicts (3.13), and the theorem is proved.

Proof of Theorem IV. We consider $\Lambda(u)$ for $u > N + 1$. From (3.02),

$$\Lambda(u) \geqq 1 + 2\left[u - \tfrac{1}{2}N - \frac{p-1}{2p} \right], \qquad u > N + 1,$$

where $[x]$ denotes the largest integer not exceeding x. Thus

$$\begin{aligned} \int_{N+1}^{v} \frac{\Lambda(u)}{u}\, du &\geqq \int_{N+1}^{v} \frac{1 + 2[u - \frac{1}{2}N - (p-1)/2p]}{u}\, du \\ &= 2 \int_{N+1}^{v} \frac{u - \frac{1}{2}N - (p-1)/2p}{u}\, du \\ &\quad - 2 \int_{N+1}^{v} \frac{u - \frac{1}{2}N - (p-1)/2p - [u - \frac{1}{2}N - (p-1)/2p] - \frac{1}{2}}{u}\, du. \end{aligned}$$

Since $x - [x] - \frac{1}{2}$ is periodic and has average value zero over each period, an integration by parts will show that the second term on the right in the above inequality is bounded as $v \to \infty$. Thus

$$\int_{N+1}^{v} \frac{\Lambda(u)}{u}\, du \geqq 2 \int_{N+1}^{v} \frac{u - \frac{1}{2}N - (p-1)/2p}{u}\, du - A \tag{3.14}$$
$$> 2v - N \log v - \frac{p-1}{p} \log v - A.$$

Now let us add N terms $e^{i\alpha_n x}$, $1 \leqq n \leqq N$, to $\{e^{i\lambda_n x}\}$ and denote the new set by $\{e^{i\mu_n x}\}$. If $\mu(u)$ represents the number of $| \mu_n | \leqq u$, then for large u, $\mu(u) = \Lambda(u) + N$. Thus by (3.14)

$$\int_{1}^{v} \frac{\mu(u)}{u}\, du > 2v - \frac{p-1}{p} \log v - A.$$

From Theorem III it follows that $\{e^{i\mu_n x}\}$ is closed $L^p(-\pi, \pi)$. This completes the proof.

Before proving Theorem V we shall prove Theorem VI.

Proof of Theorem VI. Let $\{e^{i\lambda_n x}\}$ be closed and let the set consisting of $\{e^{i\lambda_n x}\}$, $n \neq 0$, together with $e^{i\alpha x}$, $\alpha \neq \lambda_0$, not be closed. Then there exists an $f(x) \in L^p(-\pi, \pi)$ not equivalent to zero such that

$$\int_{-\pi}^{\pi} f(x) e^{i\lambda_n x}\, dx = 0, \quad n \neq 0, \qquad \int_{-\pi}^{\pi} f(x) e^{i\alpha x}\, dx = 0.$$

Let us consider

$$g(x) = f(x) + i(\lambda_0 - \alpha) e^{-i\alpha x} \int_{-\pi}^{x} f(y) e^{i\alpha y}\, dy.$$

Clearly $g(x) \in L^p(-\pi, \pi)$. Moreover,

$$\int_{-\pi}^{\pi} g(x) e^{iux}\, dx = \int_{-\pi}^{\pi} f(x) e^{iux}\, dx + i(\lambda_0 - \alpha) \int_{-\pi}^{\pi} e^{i(u-\alpha)x}\, dx \int_{-\pi}^{x} f(y) e^{i\alpha y}\, dy,$$

or, integrating by parts

$$\int_{-\pi}^{\pi} g(x) e^{iux}\, dx = \frac{u - \lambda_0}{u - \alpha} \int_{-\pi}^{\pi} f(x) e^{iux}\, dx. \tag{3.15}$$

It follows at once from this, on setting $u = \lambda_n$, that for any n

$$\int_{-\pi}^{\pi} g(x) e^{i\lambda_n x}\, dx = 0.$$

But $\{e^{i\lambda_n x}\}$ is closed $L^p(-\pi, \pi)$, and therefore $g(x)$ is equivalent to zero. But by (3.15) this means that

$$\int_{-\pi}^{\pi} f(x) e^{iux}\, dx = 0.$$

If we set $u = 0, \pm 1, \pm 2, \cdots$, this implies that $f(x)$ must also be equivalent to zero, contrary to our assumption.

Proof of Theorem V. First let us take the case when N is odd. We take

$$\begin{aligned} \lambda_n &= n + \tfrac{1}{2}N + \delta + (p-1)/2p, \qquad n > 0, \\ \lambda_{-n} &= -\lambda_n, \qquad n > 0, \\ \lambda_0 &= \tfrac{1}{2}N + \delta + (p-1)/2p. \end{aligned} \tag{3.16}$$

By Theorem VI it of course does not matter where we take λ_0 (or any other finite number of λ_n). Let us set $\frac{1}{2} + \delta + (p-1)/2p = t$. Then clearly $\cos^{2t-2}\frac{1}{2}x \in L^p(-\pi, \pi)$. Moreover, for $n \geqq 0$,

$$\begin{aligned} \int_{-\pi}^{\pi} e^{i(n+t)x} \cos^{2t-2} \tfrac{1}{2}x \, dx &= 2^{-2t+2} \int_{-\pi}^{\pi} e^{i(n+1)x}(1 + e^{ix})^{2t-2} \, dx \\ &= \lim_{r \to 1-0} 2^{-2t+2} \int_{-\pi}^{\pi} e^{i(n+1)x}(1 + re^{ix})^{2t-2} \, dx. \end{aligned}$$

Using the binomial theorem, we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} e^{i(n+t)x} \cos^{2t-2} \tfrac{1}{2}x \, dx &= \lim_{r \to 1-0} 2^{-2t+2} \sum_{k=0}^{\infty} r^k \binom{2t-2}{k} \int_{-\pi}^{\pi} e^{i(n+k+1)x} \, dx \\ &= 0. \end{aligned}$$

A similar result holds with $e^{-ix(n+t)}$, $n \geqq 0$. Thus $\cos^{2t-2}\frac{1}{2}x$ is orthogonal to $e^{\pm ix(n+t)}$, $n \geqq 0$. But the set $\{\pm(n+t)\}$, $(n \geqq 0)$, contains the set $\{\lambda_n\}$ defined in (3.16) and N additional terms. This proves Theorem V if N is odd.

If N is even, we proceed similarly. Instead of using $\cos^{2t-2}\frac{1}{2}x$ we use $\sin\frac{1}{2}x \cos^{2t-1}\frac{1}{2}x$, where $t = \delta + (p-1)/2p$. In this case $\{1, e^{\pm ix(n+t)}\}$, $(n \geqq 1)$, is orthogonal to $\sin\frac{1}{2}x \cos^{2t-1}\frac{1}{2}x$.

CHAPTER II

ON THE CLOSURE OF $\{e^{i\lambda_n x}\}$, II

4. A closure theorem involving Pólya maximum density. Theorems I and II give information on the closure of $\{e^{i\lambda_n x}\}$, $(\lambda_n > 0)$. In case

$$\lim_{n \to \infty} \frac{n}{\lambda_n} = D \tag{4.01}$$

exists, Theorem I gives a sharp criterion. However if n/λ_n does not approach a limit, this is no longer the case. From Theorems I and II it is clear that the denser the λ_n's, that is, the bigger D is, the longer the interval over which $\{e^{i\lambda_n x}\}$ will be closed.

Let us consider a sequence which does not have a density in the above sense. Let $\{\mu_n\}$ consist of all integers $n > 0$ such that

$$10^k < n \leqq 10^k + 10^{k-1}, \qquad k \geqq 1.$$

Thus $\{\mu_n\}$ consists of 11; 101, 102, $\cdots$, 109, 110; 1001, $\cdots$, 1100; and so on. Clearly the number of $\mu_n \leqq 10^k$ is

$$1 + 10 + \cdots + 10^{k-2} = (10^{k-1} - 1)/9.$$

Thus

$$\liminf_{n \to \infty} \frac{n}{\mu_n} \leqq \lim_{k \to \infty} \frac{10^{k-1} - 1}{9(10^k)}$$

or

$$\liminf_{n \to \infty} \frac{n}{\mu_n} \leqq \frac{1}{90}.$$

Thus on the basis of Theorem II we might suspect that $\{e^{i\mu_n x}\}$ is closed over an interval of length at most $2\pi/90$. Actually it will turn out that this is not the case but that the set $\{\mu_n\}$ is closed over any interval of length less than 2π. That is, in some sense the effective density, so far as closure is concerned, of the set $\{\mu_n\}$ is the same as that of all the positive integers, and is therefore one. Actually the sequence $\{\mu_n\}$ does have a density one, in the intervals (100, 110); (1000, 1100); $\cdots$; $(10^k, 10^k + 10^{k-1})$; $\cdots$. That it is the density in such subintervals that determines closure properties is implied by the following theorem.

THEOREM VII.[1] *Let* $\{\lambda_n\}$, $n > 0$, *be an increasing positive sequence and let* $\Lambda(u)$ *be the number of* $\lambda_n < u$. *Let*[2]

$$\limsup_{\xi \to 1-0} \limsup_{u \to \infty} \frac{\Lambda(u) - \Lambda(u\xi)}{u - u\xi} = D. \tag{4.02}$$

Then $\{e^{i\lambda_n x}\}$ *is closed over an interval of length* L *if*

$$2\pi D > L, \tag{4.03}$$

for Lebesgue integrable functions.

Theorem VII as a closure theorem is very easily derived from the following theorem on entire functions.

THEOREM VIII.[3] *Let* $f(z)$ *be an entire function such that*

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1 + x^2} dx < \infty, \tag{4.04}$$

and

$$\limsup_{r \to \infty} \frac{\log |f(re^{i\theta})|}{r} \leqq k. \tag{4.05}$$

Let $n_1(r)$ *be the number of zeros of* $f(z)$ *which lie in the sector* ($|z| < r$, $|\text{am } z| \leqq \frac{1}{2}\pi$) *and* $n_2(r)$ *the number of zeros of* $f(z)$ *in the sector* ($|z| < r$, $\frac{1}{2}\pi < \text{am } z < \frac{3}{2}\pi$). *Then there exists a number* $B \leqq k/\pi$ *such that*

$$\lim_{r \to \infty} \frac{n_1(r)}{r} = B, \qquad \lim_{r \to \infty} \frac{n_2(r)}{r} = B. \tag{4.06}$$

The proof of Theorem VIII will be given in Chapter III as a corollary of a less restrictive theorem. Here we shall use Theorem VIII to give a proof of Theorem VII.

Proof of Theorem VII. Suppose that a set $\{e^{i\lambda_n x}\}$ is not closed. Then there exists a function $g(x)$, not equivalent to zero, such that

$$\int_{-L/2}^{L/2} g(x)e^{i\lambda_n x} dx = 0, \qquad n > 0.$$

[1] Levinson, *On the closure of* $\{e^{i\lambda_n x}\}$ *and integral functions*, Proceedings of the Cambridge Philosophical Society, vol. 31 (1935), p. 335.

[2] D is the Pólya maximum density of the sequence. Pólya shows that there exists a sequence $\{r_n\}$ of ordinary density [(4.01)] D which contains $\{\lambda_n\}$ as a subsequence but that there is no sequence having an ordinary density less than D which contains $\{\lambda_n\}$.

Pólya also shows that in (4.02) the use of lim sup as $\xi \to 1 - 0$ is unnecessary since as $\xi \to 1 - 0$ the limit itself exists. Pólya, *Untersuchungen über Lücken und Singularitäten von Potenzreihen*, Mathematische Zeitschrift, vol. 29 (1929).

[3] Levinson, loc. cit., Theorem III. For even $f(z)$ this theorem was proved by Wiener and Paley, *On entire functions*, Transactions of the American Mathematical Society, vol. 35 (1933), Theorem I, p. 769. Miss Cartwright proved this theorem under a more restrictive hypothesis which she later showed was implied by the hypothesis given here. M. L. Cartwright, Proceedings of the London Mathematical Society, (2), vol. 38 (1935), p. 179; and Proceedings of the Cambridge Philosophical Society, vol. 34 (1935), pp. 347–350.

Let

$$G(w) = \int_{-L/2}^{L/2} g(x)e^{iwx}\,dx. \tag{4.07}$$

Then $G(w)$ vanishes at $w = \lambda_n$, $(n > 0)$, but is not identically zero. By (4.07)

$$|G(re^{i\theta})| \leqq e^{(1/2)Lr|\sin\theta|} \int_{-L/2}^{L/2} |g(x)|\,dx. \tag{4.08}$$

From this it is clear that $\log^+ |G(u)|$ is bounded. Thus $G(w)$ satisfies the requirements of Theorem VIII. If $n_1(r)$ is the number of zeros of $G(w)$ in $(|w| < r, |\text{am } w| \leqq \frac{1}{2}\pi)$, then

$$\lim_{r\to\infty} \frac{n_1(r)}{r} = B.$$

Therefore for any $0 < \xi < 1$,

$$\lim_{r\to\infty} \frac{n_1(r) - n_1(r\xi)}{r - r\xi} = B. \tag{4.09}$$

But $\{\lambda_n\}$ are among the zeros of $G(w)$ in the right half-plane. Therefore

$$n_1(r) - n_1(r\xi) \geqq \Lambda(r) - \Lambda(r\xi), \qquad 0 < \xi < 1.$$

Thus by (4.09)

$$\limsup_{r\to\infty} \frac{\Lambda(r) - \Lambda(r\xi)}{r - r\xi} \leqq B.$$

And therefore the Pólya maximum density, D, of the set $\{\lambda_n\}$ satisfies

$$D \leqq B. \tag{4.10}$$

It follows from (4.08) that $k \leqq \frac{1}{2}L$. Since $B \leqq k/\pi$,

$$B \leqq \frac{L}{2\pi}.$$

In (4.10) this gives

$$2\pi D \leqq L.$$

This is contrary to the hypothesis of Theorem VII. Thus Theorem VII is proved.

5. A related gap theorem. Let $f(x) \in L(-\pi, \pi)$. Suppose that the Fourier series of $f(x)$ is such that certain terms vanish. For example, let

$$f(x) \sim \sum_{-\infty}^{\infty} a_{3n} e^{i3nx}.$$

Then $f(x)$ has period $\frac{2}{3}\pi$, and therefore if $f(x)$ vanishes over an interval of length $\frac{2}{3}\pi$ it is identically zero. That is, if only every third term differs from zero in

the Fourier expansion for $f(x)$, then $f(x)$ cannot vanish over an interval of length $\frac{3}{2}\pi$ without vanishing identically.

This suggests the following gap relation. Let

$$f(x) \sim a_0 + \sum_1^\infty (a_{m_n} e^{im_n x} + a_{-m_n} e^{-im_n x}),$$

where $\{m_n\}$ is an increasing sequence of positive integers such that

$$\lim_{n\to\infty} \frac{n}{m_n} = D.$$

Then, if $f(x)$ vanishes over an interval exceeding $2\pi D$ in length, it vanishes identically.

Actually it will turn out that there is nothing to be gained by having gaps throughout the whole series. It suffices to have gaps only in one half of the series, either for $n > 0$ or for $n < 0$. Moreover n/m_n need not approach a limit. The gap theorem we shall prove here is

THEOREM IX.[4] *Let $f(x) \in L(-\pi, \pi)$ and have the Fourier series*

$$f(x) \sim \sum_0^\infty a_n e^{inx} + \sum_1^\infty a_{-m_n} e^{-im_n x} \tag{5.01}$$

where $\{m_n\}$ is an increasing sequence of positive integers. Let $M(u)$ be the number of $m_n < u$, and let[5]

$$d = \liminf_{\xi\to 1-0} \liminf_{u\to\infty} \frac{M(u) - M(u\xi)}{u - u\xi}. \tag{5.02}$$

If $f(x)$ vanishes almost everywhere over an interval exceeding $2\pi d$ in length, it vanishes almost everywhere over $(-\pi, \pi)$.

As an example, let us define $\{m_n\}$ as made up of those integers n which are such that

$$10^k + 10^{k-1} \leqq n \leqq 10^{k+1}, \qquad k \geqq 1. \tag{5.03}$$

That is, $\{m_n\}$ contains all integers between 11 and 100, between 110 and 1000, and so on. Clearly

$$\liminf_{n\to\infty} \frac{n}{m_n} = \frac{8}{9}.$$

Thus it should seem that $f(x)$ would have to vanish over an interval at least $\frac{8}{9}(2\pi)$ in length in order that $f(x)$ be forced to vanish over the whole interval

[4] Levinson, loc. cit., Theorem II Here the theorem was stated only for $d = 0$.

[5] d is the Pólya minimum density. As was the case with Pólya maximum density, Pólya shows that lim inf as $\xi \to 1 - 0$ can be replaced by lim

$(-\pi, \pi)$. Actually, in applying Theorem IX, since there are no m_n in the intervals $(10^k, 10^k + 10^{k-1})$ it follows that

$$M(u) - M(u\xi) = 0$$

for $u = 10^k + 10^{k-1}$ and $1 > \xi > \frac{10}{11}$. Thus

$$\liminf_{u\to\infty} \frac{M(u) - M(u\xi)}{u - u\xi} = 0, \qquad \frac{10}{11} < \xi < 1.$$

Therefore

$$d = 0.$$

Thus with $\{m_n\}$ defined as in (5.03), $f(x)$ cannot vanish over any interval no matter how small without vanishing almost everywhere over $(-\pi, \pi)$. That is, instead of $\frac{8}{9}$ it turns out that any quantity greater than zero will do.

Proof of Theorem IX. Suppose that the hypothesis of Theorem IX is satisfied and that $f(x)$ vanishes almost everywhere over an interval $2\pi d + \delta$, $\delta > 0$, in length. Since $f(x)$ is periodic of period 2π and can be translated without changing the modulus of the Fourier coefficients, we can assume that $f(x)$ vanishes over the interval $(\pi - 2\pi d - \delta, \pi)$. Let $\{\lambda_n\}$ be the set of positive integers complementary to $\{m_n\}$. That is, $\{\lambda_n\} + \{m_n\} = \{n\}$, $n > 0$. Thus

$$u - u\xi + 2 > \Lambda(u) - \Lambda(u\xi) + M(u) - M(u\xi) > u - u\xi - 2,$$

and therefore

$$\limsup_{u\to\infty} \frac{\Lambda(u) - \Lambda(u\xi)}{u - u\xi} + \liminf_{u\to\infty} \frac{M(u) - M(u\xi)}{u - u\xi} = 1.$$

From this it follows at once that the maximum density of $\{\lambda_n\}$ is

$$D = 1 - d.$$

Since the terms $e^{i\lambda_n x}$ do not appear in the Fourier series of $f(x)$, it follows that

$$\int_{-\pi}^{\pi} f(x) e^{i\lambda_n x}\, dx = 0, \qquad n > 0.$$

But $f(x)$ vanishes almost everywhere over $(\pi - 2\pi d - \delta, \pi)$. Thus

$$\int_{-\pi}^{\pi - 2\pi d - \delta} f(x) e^{i\lambda_n x}\, dx = 0, \qquad n > 0. \tag{5.04}$$

But $\pi - 2\pi d - \delta - (-\pi) = 2\pi(1 - d) - \delta = 2\pi D - \delta$. By Theorem VII, $\{e^{i\lambda_n x}\}$ is closed over any interval of length less than $2\pi D$. Thus (5.04) implies that $f(x)$ is equivalent to zero over $(-\pi, \pi - 2\pi d - \delta)$. This completes the proof of Theorem IX.

6. An extension of the closure theorem. Theorem VII states that if $\{\lambda_n\}$ is a positive increasing sequence with Pólya maximum density D and if $L < 2\pi D$, then

$$\int_{-L/2}^{L/2} f(x)e^{i\lambda_n x}\,dx = 0, \qquad 0 < n < \infty, \tag{6.01}$$

implies that $f(x)$ is zero almost everywhere in $(-\frac{1}{2}L, \frac{1}{2}L)$.

This result remains valid if instead of requiring that the integral in (6.01) be zero, we merely require that it be small. This is stated in precise terms in the following theorem.

THEOREM X.[6] *If $\{\lambda_n\}$ is an increasing positive sequence with Pólya maximum density D, if, for some $c > 0$, $\lambda_{n+1} - \lambda_n \geqq c$, and if*

$$2\pi D > L, \tag{6.02}$$

then

$$\int_{-L/2}^{L/2} f(x)e^{i\lambda_n x}\,dx = O(e^{-\delta\lambda_n}), \qquad 0 < n < \infty, \tag{6.03}$$

for some $\delta > 0$ implies that $f(x)$ is zero almost everywhere.

Theorem X is an immediate consequence of the following theorem on entire functions.

THEOREM XI.[7] *Let $f(z)$ be an entire function satisfying* (4.04) *and* (4.05). *Let $\{\lambda_n\}$ be an increasing positive sequence such that for some $c > 0$, $\lambda_{n+1} - \lambda_n \geqq c$, and let the Pólya maximum density of $\{\lambda_n\}$ be D. If*

$$\pi D > k, \tag{6.04}$$

and if, for some $\delta > 0$,

$$f(\lambda_n) = O(e^{-\delta\lambda_n}), \qquad n > 0, \tag{6.05}$$

then $f(z) \equiv 0$.

The proof of Theorem XI will depend on Theorem VIII and will be given after the proof of Theorem VIII. We shall make use of Theorem XI at once to prove Theorem X.

Proof of Theorem X. Let us assume that Theorem X is not true. Then there exists a sequence $\{\lambda_n\}$ and a function $f(x)$ not equivalent to zero and satisfying the requirements of Theorem X. That is,

$$\int_{-L/2}^{L/2} f(x)e^{i\lambda_n x}\,dx = O(e^{-\delta\lambda_n}). \tag{6.06}$$

Let

[6] Cf. Levinson, *On the non-vanishing of certain functions*, Proceedings of the National Academy of Sciences, vol. 22 (1936), p. 228, Theorem III.

[7] This theorem is referred to on page 229, Levinson, loc cit. At about the same time an even more general result was given by Miss M L. Cartwright, Proceedings of the London Mathematical Society, vol. 41 (1936), p. 33.

(6.07) $$F(w) = \int_{-L/2}^{L/2} f(x)e^{iwx}\,dx.$$

Then if $w = u + iv$,

(6.08) $$|F(w)| \leqq e^{L|v|/2} \int_{-L/2}^{L/2} |f(x)|\,dx.$$

Thus $F(w)$ satisfies the hypothesis of Theorem VIII and therefore part of the hypothesis of Theorem XI. From (6.06),

(6.09) $$F(\lambda_n) = O(e^{-\delta\lambda_n}), \qquad n \to \infty.$$

Also from (6.08)

$$\limsup_{|w|\to\infty} \frac{\log |F(w)|}{|w|} \leqq \tfrac{1}{2}L = k.$$

By (6.02), $\frac{1}{2}L < \pi D$. Thus

(6.10) $$k < \pi D.$$

But (6.09) and (6.10), used in Theorem XI, shows that the entire function $F(w)$ is zero. By (6.07) this means that $f(x)$ is equivalent to zero contrary to hypothesis. This completes the proof of Theorem X.

As a theorem on closure, Theorem X can be used to deduce a gap theorem which is really a generalization of Theorem IX.

THEOREM XII. *Let*

$$f(x) \sim \sum_{-\infty}^{\infty} a_n e^{inx}.$$

Let $\{\lambda_n\}$ *be an increasing sequence of positive integers of Pólya maximum density* D. *If*

(6.11) $$a_{-\lambda_n} = O(e^{-\delta\lambda_n}), \qquad n > 0,$$

and if $f(x)$ *vanishes almost everywhere over an interval exceeding* $2\pi(1 - D)$ *in length, then* $f(x)$ *vanishes almost everywhere.*

Theorem IX is the special case of Theorem XII where (6.11) is replaced by $a_{-\lambda_n} = 0$.

Proof of Theorem XII. Since $f(x)$ is periodic, we can assume it vanishes almost everywhere over the interval $(\pi - 2\pi(1 - D) - \epsilon, \pi)$ for some $\epsilon > 0$. Clearly then

$$a_{-\lambda_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi - 2\pi(1-D)-\epsilon} f(x)e^{i\lambda_n x}\,dx.$$

But $a_{-\lambda_n} = O(e^{-\delta\lambda_n})$. Thus

(6.12) $$\int_{-\pi}^{\pi-2\pi(1-D)-\epsilon} f(x)e^{i\lambda_n x}\,dx = O(e^{-\delta\lambda_n}).$$

The length of the interval of integration in (6.12) is $L = 2\pi D - \epsilon$. That is,

$$L < 2\pi D.$$

Applying Theorem X, it follows that $f(x)$ must vanish almost everywhere.

7. Another type of theorem. There is another closure condition which, in the sense that it deals with an ordinary density (not a Pólya maximum or minimum density), is a generalization of a result of Chapter I. Actually, however, the proof of the theorem places it here.

THEOREM XIII.[8] *Let $\{\lambda_n\}$, $(n > 0)$, be an increasing positive sequence such that*

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D, \tag{7.01}$$

and such that for some $c > 0$, $\lambda_{n+1} - \lambda_n \geqq c$. Let

$$L < 2\pi D, \tag{7.02}$$

and $\theta(u)$ be a monotone increasing function of u such that

$$\int_1^\infty \frac{\theta(u)}{u^2}\,du = \infty. \tag{7.03}$$

If

$$\int_{-L/2}^{L/2} f(x)e^{i\lambda_n x}\,dx = O(e^{-\theta(\lambda_n)}), \qquad n > 0, \tag{7.04}$$

then $f(x)$ is zero almost everywhere,

If the integral in (7.04) vanished instead of being small, then Theorem XIII would be a corollary of Theorem II. However, actually, this theorem is very different from Theorem II. Theorem XIII is a consequence of the following theorem on entire functions.

THEOREM XIV.[9] *Let $f(z)$ be an entire function satisfying the requirements of Theorem VIII, (4.04) and (4.05). Let $\{\lambda_n\}$ be an increasing positive sequence such that for some $c > 0$, $\lambda_{n+1} - \lambda_n \geqq c$, and*

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D. \tag{7.05}$$

Let

$$\pi D > k. \tag{7.06}$$

If

[8] Cf. Levinson, loc. cit., Theorem II.

[9] This theorem is referred to on page 229, Levinson, loc. cit., and is a little less restrictive than a corresponding result of Miss Cartwright, loc. cit.

$$f(\lambda_n) = O(e^{\theta(\lambda_n)}) \tag{7.07}$$

where $\theta(u)$ is a monotone increasing function of u such that

$$\int_1^\infty \frac{\theta(u)}{u^2}\,du = \infty, \tag{7.08}$$

then $f(z)$ is zero.

The proof of Theorem XIII follows from Theorem XIV in the same way as other theorems on closure in this chapter have followed from theorems on entire functions.

From Theorem XIII there follows the next gap theorem:

THEOREM XV. *Let*

$$f(x) \sim \sum_{-\infty}^{\infty} a_n e^{inx}.$$

Let $\{\lambda_n\}$ be an increasing sequence of positive integers such that

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D.$$

Let

$$a_{-\lambda_n} = O(e^{-\theta(\lambda_n)}), \tag{7.09}$$

where $\theta(u)$ is monotone increasing and

$$\int_1^\infty \frac{\theta(u)}{u^2}\,du = \infty.$$

If $f(x)$ vanishes almost everywhere over an interval exceeding $2\pi(1 - D)$ in length, then $f(x)$ is equivalent to zero.

This theorem follows from Theorem XIII in the way that Theorem XII follows from Theorem X.

8. A theorem of Miss Cartwright. The gap theorems given above can be put into more general form.[10]

If we let

$$h_1(z) = \sum_0^\infty a_n z^n, \qquad |z| < 1,$$

$$h_2(z) = -\sum_1^\infty \frac{a_{-n}}{z^n}, \qquad |z| > 1,$$

where (7.09) is satisfied, then heuristically, at least,

$$f(x) = \lim_{r\to 1} \{h_1(re^{ix}) - h_2(re^{-ix})\}$$

[10] Cf. Miss Cartwright, loc. cit.

is the $f(x)$ of Theorem XV. For $f(x)$ to vanish over an interval of length exceeding $2\pi(1 - D)$ in length amounts to $h_1(z) = h_2(z)$ over an arc of the unit circle exceeding $2\pi(1 - D)$ in length. But under very general conditions, for $h_1(z)$ to be equal to $h_2(z)$ over the arc on the unit circle means that there exists a function $H(z)$ such that $H(z)$ is analytic for $|z| < 1$ and $|z| > 1$;

$$H(z) = h_1(z),\ |z| < 1, \qquad H(z) = h_2(z),\ |z| > 1;$$

and $H(z)$ is analytic on the arc of the unit circle exceeding $2\pi(1 - D)$ in length.

The conclusion of Theorem XV that $f(x)$ vanishes entirely means here that $H(z)$ is analytic on the whole circumference of the unit circle. Combining this with the previous information on $H(z)$ gives the result that $H(z)$ is analytic on the whole plane including ∞. Thus $H(z)$ is a constant. If account is taken of the series representation of $h_2(z)$, it follows that this constant must be zero. Thus $H(z)$ is zero.

This result is now given in precise terms.

THEOREM XVI. *Let*

$$H(z) = \sum_0^\infty a_n z^n, \qquad |z| < 1,$$

$$H(z) = \sum_1^\infty a_{-n} z^{-n}, \qquad |z| > 1.$$

Also let $H(z)$ be analytic for $(|z| = 1, |\operatorname{am} z| \leqq \Theta)$. Let $\{\lambda_n\}$ be a sequence of integers such that

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D.$$

Let

(8.01) $$a_{-\lambda_n} = O(e^{-\theta(\lambda_n)})$$

where $\theta(u)$ is monotone increasing and

$$\int_1^\infty \frac{\theta(u)}{u^2}\,du = \infty.$$

Let

(8.02) $$|H(re^{i\theta})| \leqq M(r - 1), \qquad \tfrac{1}{2} < r < \tfrac{3}{2},$$

where $M(u)$ is a positive even function, decreasing for $u > 0$, and

(8.03) $$\int_0^{1/2} \log\log M(u)\,du < \infty.$$

Then if

(8.04) $$\Theta > \pi(1 - D),$$

$H(z)$ is zero.

Proof of Theorem XVI. Let

$$F(w) = \frac{e^{\pi i w}}{2\pi i}\int_C \frac{H(z)}{z^{w+1}}\,dz \tag{8.05}$$

where C is the path shown in Fig. 1. We can deform C into C_1. The two parts of C_1, consisting of circular arcs with O as center, have radii r_1 and r_2, with r_1

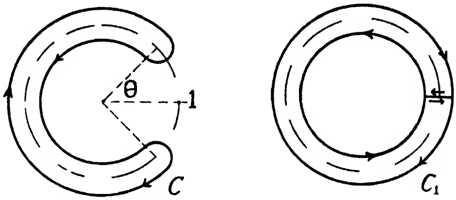

FIG 1

the larger radius and $r_1 - 1 = 1 - r_2$. Then clearly by (8.02), for $u > 0$, (8.05) with C_1 in place of C gives

$$\log^+ |F(u)| \leqq \log 4\pi M(r_1 - 1) + u \log 1/r_2 + \log^+\left(2\int_{r_2}^{r_1} \frac{|H(x)|}{x^{u+1}}\,dx\right).$$

Since $H(z)$ is analytic along the real axis, there exists a constant A such that

$$\log^+ |F(u)| \leqq \log M(r_1 - 1) + 2u \log \frac{1}{r_2} + A.$$

Let $r_1 - 1 = t$. Then

$$\log^+ |F(u)| \leqq \log M(t) + 2u \log \frac{1}{1-t} + A. \tag{8.06}$$

We shall find it advantageous to take C_1 narrower, that is, t smaller, as u gets larger. Since $M(t)$ is decreasing for $t > 0$,

$$\frac{\log M(t)}{\log 1/(1-t)}, \qquad 0 < t < \tfrac{1}{2},$$

is a decreasing function of t and is infinite when t is zero; it follows that there exists a $u_0 > 0$ such that

$$u = \frac{\log M(t)}{\log 1/(1-t)} \tag{8.07}$$

is uniquely solvable for t for $u > u_0$. This implies that, for small values of t, t is a function of u.

By (8.06)

$$(8.08)\qquad \int_{u_0}^{\infty} \frac{\log^+ |F(u)|}{u^2}\, du \leqq \int_{u_0}^{\infty} \left\{ \log M(t) + 2u \log \frac{1}{1-t} \right\} \frac{du}{u^2} + \frac{A}{u_0}.$$

There is no loss in generality in assuming that $M(t)$ is differentiable. Using (8.07) to replace u by t on the right side of (8.08) gives, if t_0 corresponds to u_0,

$$\int_{u_0}^{\infty} \frac{\log^+ |F(u)|}{u^2}\, du$$

$$\leqq -\int_0^{t_0} 3 \log M(t) \frac{\log\left(\frac{1}{1-t}\right)\frac{M'(t)}{M(t)} - \frac{1}{1-t}\log M(t)}{\log^2 M(t)}\, dt + \frac{A}{u_0},$$

or

$$(8.09)\qquad \begin{aligned} \int_{u_0}^{\infty} \frac{\log^+ |F(u)|}{u^2}\, du &\leqq -3 \int_0^{t_0} \frac{M'(t) \log 1/(1-t)}{M(t) \log M(t)}\, dt + 3 \int_0^{t_0} \frac{dt}{1-t} + \frac{A}{u_0} \\ &\leqq -3 \int_0^{1/2} \frac{M'(t) \log 1/(1-t)}{M(t) \log M(t)}\, dt + \frac{A}{u_0} + 10. \end{aligned}$$

Integrating by parts gives

$$(8.10)\qquad \begin{aligned} &-\int \frac{M'(t) \log 1/(1-t)}{M(t) \log M(t)}\, dt \\ &\qquad = -\log \frac{1}{1-t} \log\log M(t) + \int \frac{\log\log M(t)}{1-t}\, dt. \end{aligned}$$

But as $t \to +0$,

$$\log \frac{1}{1-t} \log\log M(t) < 2t \log\log M(t).$$

Since $\log \log M(t)$ is integrable, there must be a sequence of values of $t \to 0$ such that for these values $t \log \log M(t) \to 0$. Thus (8.10) becomes

$$-\int_0^{1/2} \frac{M'(t) \log 1/(1-t)}{M(t) \log M(t)}\, dt = -\log 2 \log\log M(1/2) + \int_0^{1/2} \frac{\log\log M(t)}{1-t}\, dt$$

$$< 2 \int_0^{1/2} \log\log M(t)\, dt < \infty,$$

and (8.09) becomes

$$\int_1^{\infty} \frac{\log^+ |F(u)|}{u^2}\, du < \infty.$$

This result holds in the same way for $u < 0$. Thus

$$(8.11)\qquad \int_{-\infty}^{\infty} \frac{\log^+ |F(u)|}{1+u^2}\, du < \infty.$$

From (8.05), since C can be chosen so that, on C, $|\operatorname{am} z| > \Theta - \epsilon$ for any $\epsilon > 0$, it follows that

$$|F(u + iv)| \leqq e^{|v|(\pi-\Theta+\epsilon)} \int_C |H(z)| \left(\frac{1}{|z|}\right)^{u+1} |dz|.$$

And C can be chosen so that $|z|$ is close enough to 1 in value along C that

$$\left(\frac{1}{|z|}\right)^u < e^{\epsilon|u|}.$$

Thus

$$|F(u + iv)| \leqq e^{|v|(\pi-\Theta+\epsilon)+|u|\epsilon} \int_C \left|\frac{H(z)}{z} dz\right|.$$

From this it follows that for some A

$$\log |F(w)| \leqq (\pi - \Theta + 2\epsilon) |w| + A.$$

Or

$$\limsup_{|w|\to\infty} \frac{\log |F(w)|}{|w|} \leqq \pi - \Theta + 2\epsilon.$$

Since ϵ is arbitrary,

$$\limsup_{|w|\to\infty} \frac{\log |F(w)|}{|w|} \leqq \pi - \Theta = k. \tag{8.12}$$

If w is a negative integer, the path of integration in (8.05) can be deformed into a circle of radius greater than one. Then from the formula for the Laurent coefficients of $H(z)$ it follows that

$$|a_{-n}| = |F(-n)|, \qquad n > 0.$$

Thus by (8.01)

$$F(-\lambda_n) = O(e^{-\theta(\lambda_n)}). \tag{8.13}$$

Since $\pi(1 - D) < \Theta$ it follows that

$$k = \pi - \Theta < \pi D.$$

This last result together with (8.11), (8.12), and (8.13) used in Theorem XIV gives $F(w) \equiv 0$. Thus $a_{-n} = 0$, $(n > 0)$, and therefore $H(z) = 0$.

CHAPTER III

ZEROS OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

9. The basic theorem. In this chapter we shall prove the theorems on entire functions used in obtaining the gap and closure theorems in the preceding chapter. These theorems on entire functions are of considerable interest in themselves. Roughly stated these theorems assert that if an entire function $f(z)$ satisfies

$$\limsup_{|z|\to\infty} \frac{\log|f(z)|}{|z|} = k, \qquad 0 < k < \infty,$$

and if $f(x)$ is of less than exponential growth along the real axis, then $f(z)$ is similar to $\sin kz$ in many respects. Here we are especially interested in the fact that the zeros of $f(z)$ have much in common with the zeros of $\sin kz$. Thus most of the zeros of $f(z)$ stay close to the real axis. The zeros in each half-plane, $x > 0$ and $x < 0$, have density k/π which is, of course, the case with $\sin kz$. That the zeros of $f(z)$ must cluster about the real axis will always be a simple consequence of Carleman's theorem, Theorem B.

Our technique in proving these theorems will be to show that the theorems are true if they hold for functions $f(z)$ having only real zeros. It is possible to give a real-variable proof for the case where $f(z)$ has real zeros. However here a mixture of real and complex variable methods will be used.

In view of the importance of these theorems in themselves, we shall prove our basic theorem under a more general hypothesis than that given in the last chapter.

THEOREM XVII. *Let $f(z)$ be an entire function such that*

$$\lim_{R\to\infty} \int_1^R \log|f(x)f(-x)|\frac{dx}{x^2} \tag{9.01}$$

exists and is finite. Let

$$\limsup_{x\to\infty} \frac{\log|f(\pm x)|}{x} \leqq 0 \tag{9.02}$$

and

$$\limsup_{|z|\to\infty} \frac{\log|f(z)|}{|z|} \leqq k. \tag{9.03}$$

Let the function $n_1(r)$ be the number of zeros of $f(z)$ less than r in modulus which lie in the right half-plane and $n_2(r)$ the number of zeros in the left half-plane. Then there exists

$$(9.04) \qquad \lim_{r\to\infty} \frac{n_1(r)}{r} = B, \qquad \lim_{r\to\infty} \frac{n_2(r)}{r} = B,$$

and

$$B \leqq k/\pi.$$

First we shall use this theorem to prove Theorem VIII. Theorem VIII differs from Theorem XVII in that (9.01) and (9.02) are replaced in Theorem VIII by

$$(9.05) \qquad \int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2}\, dx < \infty.$$

Proof of Theorem VIII. We shall show that (9.05) and (9.03) imply (9.01) and (9.02). Thus Theorem VIII will be made a corollary of Theorem XVII.

First we shall prove that (9.03) and (9.05) imply (9.01). Applying Carleman's theorem, Theorem B, to $f(z)$ in the upper half-plane, and replacing the left side of (41.03) which is always positive by zero, it follows that

$$0 \leqq \frac{1}{2\pi}\left(\int_1^R + \int_{-R}^{-1}\right)\left(\frac{1}{x^2} - \frac{1}{R^2}\right)\log|f(x)|\,dx$$
$$+ \frac{1}{\pi R}\int_0^\pi \log|f(Re^{i\theta})|\sin\theta\, d\theta + A,$$

where A will be used to represent various constants depending only on $f(z)$. Clearly

$$-\frac{1}{2\pi}\left(\int_{-R}^{-1} + \int_1^R\right)\log^-|f(x)|\left(\frac{1}{x^2} - \frac{1}{R^2}\right)dx$$
$$\leqq \frac{1}{2\pi}\left(\int_{-R}^{-1} + \int_1^R\right)\log^+|f(x)|\frac{dx}{x^2} + \frac{1}{\pi R}\int_0^\pi \log^+|f(Re^{i\theta})|\,d\theta + A.$$

Using (9.03) and (9.05) on the right, it follows that

$$-\frac{1}{2\pi}\left(\int_{-R}^{-1} + \int_1^R\right)\log\ |f(x)|\left(\frac{1}{x^2} - \frac{1}{R^2}\right)dx \leqq A.$$

From this

$$-\frac{1}{2\pi}\left(\int_{-R/2}^{-1} + \int_1^{R/2}\right)\log^-|f(x)|\left(\frac{1}{x^2} - \frac{1}{R^2}\right)dx \leqq A.$$

But for $|x| \leqq \frac{1}{2}R$,

$$\frac{1}{x^2} - \frac{1}{R^2} > \frac{1}{2x^2}.$$

Thus

$$-\frac{1}{4\pi}\left(\int_{-R/2}^{-1}+\int_{1}^{R/2}\right)\log^{-}|f(x)|\frac{dx}{x^2}\leqq A.$$

Since R can be taken arbitrarily large, it follows that

$$-\left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)\log^{-}|f(x)|\frac{dx}{x^2}<\infty.$$

Combining this result with (9.05) gives

$$\int_{-\infty}^{\infty}\frac{|\log|f(x)||}{1+x^2}\,dx<\infty, \tag{9.06}$$

which obviously implies (9.01).

Next we shall show that (9.03) and (9.05) imply (9.02). Increasing the right side of (41.08) in Theorem H, it follows that

$$\begin{aligned}\frac{\log|f(re^{i\theta})|}{r\sin\theta}\leqq{}&\frac{1}{\pi}\int_{-R}^{R}\frac{\log^{+}|f(x)|}{x^2-2rx\cos\theta+r^2}\,dx\\&+\frac{1}{\pi}\int_{-R}^{R}\frac{R^2|\log|f(x)||}{R^4-2rxR^2\cos\theta+r^2x^2}\,dx\\&+\frac{2R}{\pi}\int_{0}^{\pi}\log^{+}|f(Re^{i\phi})|\frac{\sin\phi(R^2-r^2)}{|R^2e^{2i\phi}-2rR\cos\theta\, e^{i\phi}+r^2|^2}\,d\phi.\end{aligned}$$

Or, letting $R\to\infty$,

$$\begin{aligned}\frac{\log|f(re^{i\theta})|}{r\sin\theta}\leqq{}&\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log^{+}|f(x)|}{x^2-2rx\cos\theta+r^2}\,dx\\&+\limsup_{R\to\infty}\frac{1}{\pi R^2}\int_{-R}^{R}|\log|f(x)||\,dx\\&+\limsup_{R\to\infty}\frac{2}{\pi R}\int_{0}^{\pi}\log^{+}|f(Re^{i\phi})|\sin\phi\,d\phi.\end{aligned}\tag{9.07}$$

But since $1/R^2<2/(1+x^2)$ for $|x|<R$, $R>1$,

$$\limsup_{R\to\infty}\frac{1}{R^2}\int_{-R}^{R}|\log|f(x)||\,dx<\limsup_{R\to\infty}\int_{-R}^{R}\frac{2|\log|f(x)||}{1+x^2}\,dx<\infty$$

by (9.06). By (9.03),

$$\limsup_{R\to\infty}\frac{2}{\pi R}\int_{0}^{\pi}\log^{+}|f(Re^{i\phi})|\sin\phi\,d\phi\leqq 2k.$$

Thus (9.07) becomes

$$\frac{\log|f(re^{i\theta})|}{r\sin\theta}\leqq\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\log^{+}|f(x)|}{x^2-2rx\cos\theta+r^2}\,dx+A. \tag{9.08}$$

But

$$\int_1^\infty \frac{\log^+ |f(x)|}{x^2 - 2rx\cos\theta + r^2}\,dx < \int_1^{r/2} \frac{\log^+ |f(x)|}{(r-x)^2}\,dx + \int_{r/2}^{3r/2} \frac{\log^+ |f(x)|}{(x - r\cos\theta)^2 + r^2\sin^2\theta}\,dx + \int_{3r/2}^\infty \frac{\log^+ |f(x)|}{(x-r)^2}\,dx$$

$$< \int_1^{r/2} \frac{\log^+ |f(x)|}{x^2}\,dx + \int_{r/2}^{3r/2} \frac{\log^+ |f(x)|}{r^2\sin^2\theta}\,dx + 4\int_{3r/2}^\infty \frac{\log^+ |f(x)|}{x^2}\,dx$$

$$< 4\int_1^\infty \frac{\log^+ |f(x)|}{x^2}\,dx + \frac{4}{\sin^2\theta}\int_{r/2}^{3r/2} \frac{\log^+ |f(x)|}{x^2}\,dx.$$

For any fixed θ, $(0 < \theta < \pi)$, it follows that

$$\limsup_{r\to\infty} \int_1^\infty \frac{\log^+ |f(x)|}{x^2 - 2xr\cos\theta + r^2}\,dx \leqq 4\int_1^\infty \frac{\log^+ |f(x)|}{x^2}\,dx.$$

A similar result holds for $(-\infty, -1)$. Thus (9.08) becomes

$$\limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r|\sin\theta|} \leqq A,$$

for any fixed θ, $(0 < \theta < \pi)$. But this holds in the same way for $0 > \theta > -\pi$. Therefore for any $\delta > 0$, $f(z)e^{-\delta z}$ is bounded along the two radii $\theta = \pm\epsilon$ for a sufficiently small value of ϵ. By Theorem C of Phragmén-Lindelöf this implies that $f(z)e^{-\delta z}$ is bounded in the whole sector $|\theta| \leqq \epsilon$. Thus

$$\limsup_{x\to\infty} \frac{\log |f(x)|}{x} \leqq \delta.$$

Since δ can be taken arbitrarily small, (9.02) holds for positive x. In precisely the same way it holds for negative x.

This completes the proof that Theorem VIII is a corollary of Theorem XVII.

In proving Theorem XVII the following lemma is required. As above we shall continue to use A to represent various constants depending on $f(z)$.

Lemma 9.1. *Let the zeros of $f(z)$ be $z_n = r_n e^{i\theta_n}$. Then*

$$\sum_1^\infty \frac{|\sin\theta_n|}{r_n} < \infty.$$

Proof of Lemma 9.1. Applying Carleman's theorem, Theorem B, to $f(z)$ in the upper and lower half-planes and adding the result, we have

$$\sum_{r_n<R} \frac{|\sin\theta_n|}{r_n}\left(1 - \frac{r_n^2}{R^2}\right) \leqq \frac{1}{\pi}\int_1^R \left(\frac{1}{x^2} - \frac{1}{R^2}\right)\log|f(x)f(-x)|\,dx + \frac{1}{\pi R}\int_0^{2\pi} \log^+ |f(Re^{i\theta})|\,|\sin\theta|\,d\theta + A.$$

Using (9.03) it follows that

$$\sum_{r_n<R} \frac{|\sin\theta_n|}{r_n}\left(1-\frac{r_n^2}{R^2}\right) \leqq \frac{1}{\pi}\int_1^R \left(\frac{1}{x^2}-\frac{1}{R^2}\right)\log|f(x)f(-x)|\,dx + A.$$

Using (9.01) this becomes

$$\text{(9.09)}\qquad \sum_{r_n<R} \frac{|\sin\theta_n|}{r_n}\left(1-\frac{r_n^2}{R^2}\right) \leqq -\frac{1}{\pi R^2}\int_1^R \log|f(x)f(-x)|\,dx + A.$$

But integrating by parts we have

$$\frac{1}{R^2}\int_1^R \log|f(x)f(-x)|\,dx = \int_1^R \log|f(x)f(-x)|\frac{dx}{x^2} - \frac{1}{R^2}\int_1^R 2x\,dx\int_1^x \log|f(y)f(-y)|\frac{dy}{y^2}.$$

By (9.01)

$$\int_1^x \log|f(y)f(-y)|\frac{dy}{y^2}$$

is bounded. Therefore the integration by parts above gives

$$\left|\frac{1}{R^2}\int_1^R \log|f(x)f(-x)|\,dx\right| \leqq A + \frac{A}{R^2}\int_1^R 2x\,dx < A.$$

Thus (9.09) becomes

$$\sum_{r_n<R} \frac{|\sin\theta_n|}{r_n}\left(1-\frac{r_n^2}{R^2}\right) < A.$$

But this implies

$$\sum_{r_n<R/2} \frac{|\sin\theta_n|}{r_n}(1-\tfrac{1}{4}) < A.$$

Since R can be taken arbitrarily large above, it follows that

$$\sum_1^\infty \frac{|\sin\theta_n|}{r_n} < \infty.$$

10. A related function with real zeros. By the Hadamard factorization theorem, Theorem D,

$$f(z) = bz^m e^{az}\prod_1^\infty \left(1-\frac{z}{z_n}\right)e^{z/z_n}.$$

Clearly if Theorem XVII is proved for $f(z)/bz^m$ it is also true for $f(z)$. Thus we assume in the rest of this chapter that

$$\text{(10.01)}\qquad f(z) = e^{az}\prod_1^\infty \left(1-\frac{z}{z_n}\right)e^{z/z_n}.$$

Using the notation $z_n = r_n e^{i\theta_n}$, an auxiliary function $F(z)$ having only real zeros is now introduced:

$$F(z) = e^{(1/2)(a+a)z} \prod_1^\infty \left(1 - \frac{z \cos\theta_n}{r_n}\right) e^{(z\cos\theta_n)/r_n}. \tag{10.02}$$

We now prove a series of lemmas about $F(z)$.

LEMMA 10.1. *If*

$$\limsup_{y\to\infty} \frac{\log |F(\pm iy)|}{y} = k_1,$$

then

$$\limsup_{r\to\infty} \frac{\log |F(re^{i\theta})|}{r} \leqq k_1 |\sin\theta| \tag{10.03}$$

and

$$k_1 \leqq k$$

where k is defined in (9.03).

Proof of Lemma 10.1. Since

$$\left|1 - \frac{x\cos\theta_n}{r_n}\right| \leqq \left|1 - \frac{xe^{-i\theta_n}}{r_n}\right|,$$

it follows that

$$|F(x)| \leqq |f(x)|.$$

Thus by (9.02)

$$\limsup_{x\to\infty} \frac{\log |F(\pm x)|}{x} \leqq 0. \tag{10.04}$$

By (9.02) and (9.03) and by Theorem C′ of Phragmén-Lindelof

$$\limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r} \leqq k|\sin\theta|. \tag{10.05}$$

If $n(r)$ is the number of zeros of $f(z)$ less than r in modulus, it follows from Jensen's theorem, Theorem A, and (10.05) that

$$\limsup_{R\to\infty} \frac{1}{R}\int_0^R \frac{n(r)}{r}\,dr \leqq \frac{1}{2\pi}\int_0^{2\pi} k|\sin\theta|\,d\theta = \frac{2k}{\pi}. \tag{10.06}$$

From the definition of $F(z)$,

$$\log |F(iy)| \leqq \sum_1^\infty \log\left[\left(1 + \frac{y^2\cos^2\theta_n}{r_n^2}\right)\right]^{1/2}$$

$$\leqq \sum_1^\infty \tfrac{1}{2}\log\left(1 + \frac{y^2}{r_n^2}\right) = \frac{1}{2}\int_0^\infty \log\left(1 + \frac{y^2}{u^2}\right) dn(u).$$

Integrating the right side by parts twice, we obtain

$$\log |F(iy)| \leqq \int_0^\infty \frac{2u^2y^2}{(u^2+y^2)^2}\,du\,\frac{1}{u}\int_0^u \frac{n(r)}{r}\,dr.$$

Letting $u = v\,|\,y\,|$ this becomes

(10.07) $$\frac{\log |F(iy)|}{|y|} \leqq \int_0^\infty \frac{2v^2\,dv}{(1+v^2)^2}\,\frac{1}{v|y|}\int_0^{v|y|} \frac{n(r)}{r}\,dr.$$

Letting $|\,y\,| \to \infty$ and using (10.06) it follows that

(10.08) $$k_1 = \limsup_{y\to\infty} \frac{\log |F(\pm iy)|}{y} \leqq \frac{2k}{\pi}\int_0^\infty \frac{2v^2}{(1+v^2)^2}\,dv = k.$$

That is, $k_1 \leqq k$.

From (10.02) for large $|\,z\,|$,

$$\log |F(z)| \leqq |\,a\,|r + \sum_{r_n<2r}\left\{\log\left(1+\frac{r|\cos\theta_n|}{r_n}\right) + \frac{r|\cos\theta_n|}{r_n}\right\} + 10\sum_{r_n\geqq 2r}\frac{r^2\cos^2\theta_n}{r_n^2}$$

$$\leqq |\,a\,|r + \int_0^{2r}\left\{\log\left(1+\frac{r}{u}\right)+\frac{r}{u}\right\}dn(u) + 10\int_{2r}^\infty \frac{r^2}{u^2}\,dn(u).$$

Integrating by parts

(10.09) $$\log|F(z)| \leqq a|\,r\,| + n(2r) + \int_0^{2r}\left(\frac{r}{u+r}+\frac{r}{u}\right)\frac{n(u)}{u}\,du + 20\int_{2r}^\infty \frac{r^2}{u^2}\,\frac{n(u)}{u}\,du.$$

But

$$\frac{n(u)}{u} = \frac{n(u)}{u\log 2}\int_u^{2u}\frac{dr}{r} \leqq \frac{2}{\log 2}\,\frac{1}{2u}\int_0^{2u}\frac{n(r)}{r}\,dr.$$

From (10.06) it follows that for large u,

$$\frac{n(u)}{u} < \frac{4k}{\log 2}.$$

Thus there exists an A such that for large $|\,z\,|$ (10.09) gives

(10.10) $$\log |F(z)| \leqq Ar\log r.$$

Using (10.04), (10.08), and (10.10), and Theorem C′ of Phragmén-Lindelöf gives

$$\limsup_{r\to\infty}\frac{\log|F(re^{i\theta})|}{r} \leqq k_1|\sin\theta|.$$

This completes the proof of the lemma.

Lemma 10.2. *There exists a constant C such that*

$$\lim_{R\to\infty}\int_0^R \log|F(x)F(-x)|\frac{dx}{x^2} = C. \tag{10.11}$$

Proof of Lemma 10.2. According to the hypothesis of Theorem XVII there exists

$$\lim_{R\to\infty}\int_1^R \log|f(x)f(-x)|\frac{dx}{x^2}.$$

By (10.01) and (10.02)

$$\log\left|\frac{f(x)}{F(x)}\right| = \sum_1^\infty \log\left|\frac{r_n - xe^{-i\theta_n}}{r_n - x\cos\theta_n}\right|.$$

Thus

$$\int_0^R \log|f(x)f(-x)|\frac{dx}{x^2} = \int_0^R \log|F(x)F(-x)|\frac{dx}{x^2}$$
$$+\int_{-R}^R\frac{dx}{x^2}\sum_1^\infty \tfrac{1}{2}\log\left|\frac{r_n^2-2xr_n\cos\theta_n+x^2}{r_n^2-2xr_n\cos\theta_n+x^2\cos^2\theta_n}\right|,$$

and the proof of the lemma hinges on the existence of

$$\lim_{R\to\infty}\int_{-R}^R\frac{dx}{x^2}\sum_1^\infty \log\left|\frac{r_n^2-2xr_n\cos\theta_n+x^2}{r_n^2-2xr_n\cos\theta_n+x^2\cos^2\theta_n}\right|.$$

But since each term in the sum above is positive, the limit as $R\to\infty$ will certainly exist if

$$J = \sum_1^\infty\int_{-\infty}^\infty \log\left(\frac{r_n^2-2r_nx\cos\theta_n+x^2}{r_n^2-2r_nx\cos\theta_n+x^2\cos^2\theta_n}\right)\frac{dx}{x^2}<\infty.$$

Replace x by $r_ny/|\cos\theta_n|$, unless $\theta_n = \frac{1}{2}\pi$, in which case the integral of the logarithmic term can be computed directly and is obviously less than $10/r_n$. Thus using J' and $\sum'$ to signify the omission of $\theta_n = \frac{1}{2}\pi$, we have

$$J' = \sum_1^\infty{}' \frac{|\cos\theta_n|}{r_n}\int_{-\infty}^\infty \log\left(\frac{1-2y\cos\theta_n/|\cos\theta_n|+y^2/\cos^2\theta_n}{1-2y\cos\theta_n/|\cos\theta_n|+y^2}\right)\frac{dy}{y^2}$$
$$= \sum_1^\infty{}' \frac{|\cos\theta_n|}{r_n}\int_{-\infty}^\infty \log\left(1+\frac{y^2\tan^2\theta_n}{1-2y\cos\theta_n/|\cos\theta_n|+y^2}\right)\frac{dy}{y^2}.$$

Thus

$$J' \leqq \sum_1^\infty{}' \frac{|\cos\theta_n|}{r_n}\int_{-\infty}^\infty \log\left(1+\frac{y^2\tan^2\theta_n}{(1-|y|)^2}\right)\frac{dy}{y^2}. \tag{10.12}$$

But

$$\int_{-\infty}^{\infty} \log\left(1 + \frac{y^2 \tan^2 \theta_n}{(1-|y|)^2}\right)\frac{dy}{y^2} \leqq 2\int_0^{1/2} \log(1 + 4y^2 \tan^2 \theta_n)\frac{dy}{y^2}$$
$$+ 8\int_{1/2}^{3/2} \log\left(1 + \frac{4\tan^2\theta_n}{(1-y)^2}\right) dy$$
$$+ 2\int_{3/2}^{\infty} \log(1 + 9\tan^2\theta_n)\frac{dy}{y^2}.$$

Setting $y\,|\tan\theta_n| = x$ in the first integral on the right of the above inequality, and $(y - 1) = x\,|\tan\theta_n|$ in the second integral on the right gives

$$\int_{-\infty}^{\infty} \log\left(1 + \frac{y^2 \tan^2 \theta_n}{(1-|y|)^2}\right)\frac{dy}{y^2} \leqq 2\,|\tan\theta_n|\int_0^{\infty} \log(1 + 4x^2)\frac{dx}{x^2}$$
$$+ 8\,|\tan\theta_n|\int_{-\infty}^{\infty} \log\left(1 + \frac{4}{x^2}\right) dx$$
$$+ 2\log(1 + 9\tan^2\theta_n)$$
$$\leqq 200\,|\tan\theta_n| + 2\log(1 + 9\tan^2\theta_n).$$

But, for $u > 0$, $\log(1 + u^2) < u$, as can be verified by differentiating $u - \log(1 + u^2)$. Thus $\log(1 + 9\tan^2\theta_n) \leqq 3\,|\tan\theta_n|$ and therefore

$$\int_{-\infty}^{\infty} \log\left(1 + \frac{y^2 \tan^2 \theta_n}{(1-|y|)^2}\right)\frac{dy}{y^2} \leqq 210\,|\tan\theta_n|.$$

Using this in (10.12) and recalling that the terms with $\theta_n = \frac{1}{2}\pi$ have been discussed, it follows that

$$J \leqq 210\sum_1^{\infty}\frac{|\sin\theta_n|}{r_n}.$$

By Lemma 9.1 it follows that $J < \infty$, and this completes the proof of this lemma.

11. Lemmas on the related function. We have shown that $F(z)$ satisfies the requirements of Theorem XVII. Now we shall prove Theorem XVII for $F(z)$.

LEMMA 11.1.[1] *If k_1 is defined as in Lemma* 10.1, *then*

(11.01) $$\lim_{y\to\infty}\frac{\log|F(\pm iy)|}{y} = k_1.$$

Proof of Lemma 11.1. By Lemma 10.2,

(11.02) $$\lim_{R\to\infty}\int_0^R \log|F(x)F(-x)|\frac{dx}{x^2} = C.$$

Let the zeros of $F(z)$ be denoted by $\{x_n\}$, instead of by $r_n/\cos\theta_n$, where $|x_n|$ are increasing in magnitude. Then by (10.02)

[1] The results of this and the next lemma are due to Paley and Wiener, loc. cit., chap. 5. They use Wiener's general Tauberian theorem in their proof.

$$F(x)F(-x) = \prod_1^{\infty}\left(1 - \frac{x^2}{x_n^2}\right).$$

Thus (11.02) becomes

$$\lim_{R\to\infty}\int_0^R\left(\sum_1^{\infty}\log\left|1 - \frac{x^2}{x_n^2}\right|\right)\frac{dx}{x^2} = C.$$

Or

$$\lim_{R\to\infty}\sum_1^{\infty}\int_0^R\log\left|1 - \frac{x^2}{x_n^2}\right|\frac{dx}{x^2} = C.$$

Replacing x by $|x_n|v$,

(11.03) $$\lim_{R\to\infty}\sum_1^{\infty}\frac{1}{|x_n|}\int_0^{R/|x_n|}\log|1 - v^2|\frac{dv}{v^2} = C.$$

Let us now consider

(11.04) $$\int_0^{\infty}\frac{u\,du}{(1+u^2)^2}\frac{1}{t}\int_0^{u/t}\log|1 - v^2|\frac{dv}{v^2}$$

Integrating by parts, this becomes

$$\frac{1}{2}\int_0^{\infty}\frac{du}{u^2(1+u^2)}\log\left|1 - \frac{u^2}{t^2}\right| = \frac{1}{4}\int_{\infty}^{\infty}\log\left|1 - \frac{u^2}{t^2}\right|\frac{du}{u^2(1+u^2)}$$
$$= \Re\frac{1}{4}\int_{-\infty}^{\infty}\log\left(1 - \frac{w^2}{t^2}\right)\frac{dw}{w^2(1+w^2)}.$$

We deform the path of integration in the last integral so as to avoid $w = \pm t$ by going around these points on semicircular paths of small radius, the semicircles both lying in the half-plane above the real axis. Then closing the path of integration in the upper half-plane the pole at $u = i$ gives

$$\int_0^{\infty}\frac{u\,du}{(1+u^2)^2}\frac{1}{t}\int_0^{u/t}\log|1 - v^2|\frac{dv}{v^2} = \Re\frac{2\pi i}{4}\frac{1}{2i(i^2)}\log\left(1 + \frac{1}{t^2}\right)$$
$$= -\frac{\pi}{4}\log\left(1 + \frac{1}{t^2}\right).$$

Setting $t = |x_n|/y$,

$$\int_0^{\infty}\frac{u\,du}{(1+u^2)^2}\frac{1}{|x_n|}\int_0^{uy/|x_n|}\log|1 - v^2|\frac{dv}{v^2} = -\frac{\pi}{4y}\log\left(1 + \frac{y^2}{x_n^2}\right).$$

Adding the above equations,

(11.05) $$\sum_{n=1}^{\infty}\int_0^{\infty}\frac{u\,du}{(1+u^2)^2}\frac{1}{|x_n|}\int_0^{uy/|x_n|}\log|1 - v^2|\frac{dv}{v^2} = -\frac{\pi}{2y}\log|F(\pm iy)|.$$

Clearly since

$$\int_0^\infty \log|1 - v^2| \frac{dv}{v^2} = 0,$$

$$\int_0^a \log|1 - v^2| \frac{dv}{v^2} = \begin{cases} O(a), & a \to 0, \\ O((\log a)/a) = O(1/a^{1/2}), & a \to \infty. \end{cases}$$

Thus

$$\sum_{n=1}^\infty \int_0^\infty \frac{u\,du}{(1+u^2)^2} \frac{1}{|x_n|} \left| \int_0^{uy/|x_n|} \log|1-v^2| \frac{dv}{v^2} \right|$$
$$\leqq \sum_{n=1}^\infty \left(\int_0^{|x_n|/y} \frac{u\,du}{(1+u^2)^2} \frac{1}{|x_n|} O\left(\frac{uy}{|x_n|}\right) + \int_{|x_n|/y}^\infty \frac{u\,du}{(1+u^2)^2} \frac{1}{|x_n|} O\left(\frac{|x_n|^{1/2}}{(uy)^{1/2}}\right)\right)$$
$$= O\left(\sum_{n=1}^\infty \left\{ \frac{1}{|x_n|^2} \int_0^\infty \frac{u^2 y}{(1+u^2)^2} du + \frac{1}{(|x_n|y)^{1/2}} \int_{|x_n|/y}^\infty \frac{u^{1/2} du}{u^4} \right\}\right)$$
$$= O\left(\sum_{n=1}^\infty \left\{ \frac{y}{|x_n|^2} + \frac{y^2}{|x_n|^3} \right\}\right) < \infty.$$

Thus the order of integration and summation can be inverted in (11.05) to give

$$\int_0^\infty \frac{u\,du}{(1+u^2)^2} \sum_{n=1}^\infty \frac{1}{|x_n|} \int_0^{uy/|x_n|} \log|1-v^2| \frac{dv}{v^2} = -\frac{\pi}{2y} \log|F(\pm iy)|.$$

Letting $y \to \infty$ and using (11.03), it follows that

$$C \int_0^\infty \frac{u\,du}{(1+u^2)^2} = -\frac{\pi}{2} \lim_{y\to\infty} \frac{\log|F(\pm iy)|}{y},$$

or

$$\lim_{y\to\infty} \frac{\log|F(\pm iy)|}{y} = -\frac{C}{\pi}.$$

But by the definition of k_1 as limit superior, it follows that $k_1 = -C/\pi$. This completes the proof of the lemma.

LEMMA 11.2. *If $N(r)$ is the total number of zeros of $F(z)$ less than r in modulus, then*

(11.06) $$\lim_{r\to\infty} \frac{N(r)}{r} = \frac{2k_1}{\pi}.$$

Also

(11.07) $$\lim_{R\to\infty} \frac{1}{R} \int_0^{2\pi} |\log|F(Re^{i\theta})| - k_1 R|\sin\theta|| \, d\theta = 0.$$

Proof of Lemma 11.2. By Jensen's theorem, Theorem A,

(11.08) $$\frac{1}{R} \int_0^R \frac{N(r)}{r} dr = \frac{1}{2\pi R} \int_0^{2\pi} \log|F(Re^{i\theta})| \, d\theta.$$

Thus by (10.03),

$$\limsup_{R\to\infty} \frac{1}{R}\int_0^R \frac{N(r)}{r}\,dr \leqq \frac{1}{2\pi}\int_0^{2\pi} k_1\,|\sin\theta|\,d\theta = \frac{2k_1}{\pi}. \tag{11.09}$$

By (11.01) and

$$\log|F(iy)| = \int_0^\infty \log\left(1+\frac{y^2}{r^2}\right)^{1/2} dN(r),$$

it follows that

$$\lim_{y\to\infty}\frac{1}{2y}\int_0^\infty \log\left(1+\frac{y^2}{r^2}\right)dN(r) = k_1.$$

Integrating by parts twice this becomes

$$\lim_{y\to\infty} 2\int_0^\infty \frac{r^2 y}{(r^2+y^2)^2}\,dr\,\frac{1}{r}\int_0^r \frac{N(u)}{u}\,du = k_1.$$

Since

$$\int_0^\infty \frac{2r^2 y}{(r^2+y^2)^2}\,dr = \frac{\pi}{2},$$

it follows that

$$\lim_{y\to\infty}\int_0^\infty \frac{2r^2 y}{(r^2+y^2)^2}\,dr\left\{\frac{1}{r}\int_0^r \frac{N(u)}{u}\,du - \frac{2k_1}{\pi}\right\} = 0.$$

Thus for any small $\alpha > 0$

$$\begin{aligned}\limsup_{y\to\infty}\left(\int_0^{y(1-\alpha)} + \int_y^\infty\right)\frac{2r^2 y}{(r^2+y^2)^2}\,dr\left\{\frac{1}{r}\int_0^r \frac{N(u)}{u}\,du - \frac{2k_1}{\pi}\right\}\\ + \liminf_{y\to\infty}\int_{y(1-\alpha)}^y \frac{2r^2 y}{(r^2+y^2)^2}\,dr\left\{\frac{1}{r}\int_0^r \frac{N(u)}{u}\,du - \frac{2k_1}{\pi}\right\} = 0.\end{aligned} \tag{11.10}$$

Replacing the variable r by yv, we obtain

$$\begin{aligned}\limsup_{y\to\infty}&\left(\int_0^{y(1-\alpha)} + \int_y^\infty\right)\frac{2r^2 y}{(r^2+y^2)^2}\,dr\left\{\frac{1}{r}\int_0^r \frac{N(u)}{u}\,du - \frac{2k_1}{\pi}\right\}\\ &\leqq \left(\int_0^{1-\alpha} + \int_1^\infty\right)\frac{2v^2}{(v^2+1)^2}\,dv\,\limsup_{y\to\infty}\left\{\frac{1}{vy}\int_0^{vy} \frac{N(u)}{u}\,du - \frac{2k_1}{\pi}\right\}\\ &\leqq 0,\end{aligned}$$

where this last statement follows from (11.09). Using this in (11.10)

$$\liminf_{y\to\infty}\int_{y(1-\alpha)}^y \frac{2r^2 y}{(r^2+y^2)^2}\,dr\left\{\frac{1}{r}\int_0^r \frac{N(u)}{u}\,du - \frac{2k_1}{\pi}\right\} \geqq 0.$$

Since $N(u)$ is positive and since $1/r$ is a decreasing function,

$$\liminf_{y\to\infty}\int_{y(1-\alpha)}^y \frac{2r^2 y}{(r^2+y^2)^2}\,dr\left\{\frac{1}{y(1-\alpha)}\int_0^y \frac{N(u)}{u}\,du - \frac{2k_1}{\pi}\right\} \geqq 0.$$

Replacing the variable r by yv and observing that the terms in the braces are independent of r,

$$\liminf_{y\to\infty}\left\{\frac{1}{y(1-\alpha)}\int_0^y \frac{N(u)}{u}\,du - \frac{2k_1}{\pi}\right\}\int_{1-\alpha}^1 \frac{2v^2}{(1+v^2)^2}\,dv \geqq 0.$$

Since the integral in v is positive and independent of y, it follows that the limit inferior of the term in braces is positive. Therefore

$$\liminf_{y\to\infty}\frac{1}{y}\int_0^y \frac{N(u)}{u}\,du \geqq \frac{2k_1}{\pi}(1-\alpha).$$

Since this holds for arbitrarily small $\alpha > 0$,

$$\liminf_{y\to\infty}\frac{1}{y}\int_0^y \frac{N(u)}{u}\,du \geqq \frac{2k_1}{\pi}.$$

This result and (11.09) give

$$\text{(11.11)} \qquad \lim_{R\to\infty}\frac{1}{R}\int_0^R \frac{N(r)}{r}\,dr = \frac{2k_1}{\pi}.$$

Using this in (11.08), we have

$$\frac{2k_1}{\pi} = \lim_{R\to\infty}\frac{1}{2\pi R}\int_0^{2\pi} \log|F(Re^{i\theta})|\,d\theta.$$

Or

$$\lim_{R\to\infty}\frac{1}{2\pi R}\int_0^{2\pi}\{k_1R|\sin\theta| - \log|F(Re^{i\theta})|\}\,d\theta = 0.$$

For any $\delta > 0$,

$$\text{(11.12)} \qquad \lim_{R\to\infty}\frac{1}{2\pi R}\int_0^{2\pi}\{\delta R + k_1R|\sin\theta| - \log|F(Re^{i\theta})|\}\,d\theta = \delta.$$

But by (10.03) for sufficiently large R

$$\delta R + k_1R|\sin\theta| - \log|F(Re^{i\theta})| > 0.$$

Thus (11.12) can be written as

$$\text{(11.13)} \qquad \lim_{R\to\infty}\frac{1}{2\pi R}\int_0^{2\pi}\big|\delta R + k_1R|\sin\theta| - \log|F(Re^{i\theta})|\big|\,d\theta = \delta.$$

But

$$\big|k_1R|\sin\theta| - \log|F(Re^{i\theta})|\big| \leqq \delta R + \big|\delta R + k_1R|\sin\theta| - \log|F(Re^{i\theta})|\big|.$$

Integrating the above and using (11.13), it follows that

$$\limsup_{R\to\infty}\frac{1}{2\pi R}\int_0^{2\pi}\big|k_1R|\sin\theta| - \log|F(Re^{i\theta})|\big|\,d\theta \leqq 2\delta.$$

Since δ can be taken arbitrarily small, this gives (11.07).

To get (11.06) we observe that (11.11) gives, for any $\alpha > 0$,

$$\lim_{R\to\infty} \frac{1}{R}\int_0^{R(1+\alpha)} \frac{N(r)}{r}\,dr = \frac{2k_1}{\pi}(1+\alpha).$$

Subtracting (11.11) from this gives

$$\lim_{R\to\infty} \frac{1}{R}\int_R^{R(1+\alpha)} \frac{N(r)}{r}\,dr = \frac{2k_1}{\pi}\alpha.$$

Since $N(r)$ is an increasing function and $1/r$ decreasing, it follows that

$$\limsup_{R\to\infty} \frac{1}{R}\frac{N(R)}{R(1+\alpha)}\int_R^{R(1+\alpha)} dr \leqq \frac{2k_1}{\pi}\alpha,$$

or

$$\limsup_{R\to\infty} \frac{N(R)}{R} \leqq \frac{2k_1}{\pi}(1+\alpha).$$

Since α can be taken arbitrarily small, it follows that

$$\limsup_{R\to\infty} \frac{N(R)}{R} \leqq \frac{2k_1}{\pi}.$$

Similarly

$$\liminf_{R\to\infty} \frac{N(R)}{R} \geqq \frac{2k_1}{\pi},$$

thus giving (11.06) and completing the proof of the lemma.

12. Proof of the basic theorem. In the next lemma we prove Theorem XVII for $F(z)$, that is, for functions with real zeros.

LEMMA 12.1. *If $N_1(r)$ and $N_2(r)$ are the number of zeros of $F(z)$ less than r in modulus in the right and left half-planes respectively, then*

(12.01) $$\lim_{r\to\infty} \frac{N_1(r)}{r} = \lim_{r\to\infty} \frac{N_2(r)}{r} = \frac{k_1}{\pi}.$$

Proof of Lemma 12.1. By (11.07)

(12.02) $$\lim_{R\to\infty} \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \log|F(Re^{i\theta})|\cos\theta\,d\theta = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} k_1|\sin\theta|\cos\theta\,d\theta = \frac{k_1}{\pi}.$$

Applying Carleman's theorem, Theorem B′, (41.04), to $F(z)$ in the right half-plane, we obtain

$$\int_0^R \left(\frac{1}{r} - \frac{r}{R^2}\right) dN_1(r) = \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \log|F(Re^{i\theta})|\cos\theta\,d\theta$$
$$+ \frac{1}{2\pi}\int_0^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right)\log|F(iy)F(-iy)|\,dy + \frac{F'(0) + \overline{F'(0)}}{4}.$$

If we write the above equation for $R(1+\alpha)$, $\alpha > 0$, subtract the equations, and use (12.02),

$$\lim_{R\to\infty}\left\{\int_R^{R(1+\alpha)}\frac{dN_1(r)}{r} - \int_0^{R(1+\alpha)}\frac{r}{R^2(1+\alpha)^2}dN_1(r) + \int_0^R \frac{r}{R^2}dN_1(r)\right\}$$
$$= \lim_{R\to\infty}\frac{1}{2\pi}\left\{\int_R^{R(1+\alpha)}\log|F(iy)F(-iy)|\frac{dy}{y^2} - \frac{1}{R^2(1+\alpha)^2}\int_0^{R(1+\alpha)}\log|F(iy)F(-iy)|\,dy + \frac{1}{R^2}\int_0^R \log|F(iy)F(-iy)|\,dy\right\}.$$

But by (11.01) $\log|F(iy)F(-iy)| \sim 2k_1 y$. Thus the right side of the above equation becomes $k_1 \log(1+\alpha)/\pi$. Therefore

$$\lim_{R\to\infty}\left\{\int_R^{R(1+\alpha)}\frac{dN_1(r)}{r} - \int_0^{R(1+\alpha)}\frac{r}{R^2(1+\alpha)^2}dN_1(r) + \int_0^R \frac{r}{R^2}dN_1(r)\right\} = \frac{k_1}{\pi}\log(1+\alpha).$$

Or

$$\lim_{R\to\infty}\left\{\int_R^{R(1+\alpha)}\left(\frac{1}{r} - \frac{r}{R^2(1+\alpha)^2}\right)dN_1(r) + \int_0^R \frac{r}{R^2}\left(1 - 2\alpha - \frac{1}{(1+\alpha)^2}\right)dN_1(r) + \frac{2\alpha}{R^2}\int_0^R r\,dN_1(r)\right\} = \frac{k_1}{\pi}\log(1+\alpha).$$

From this, dividing by α,

$$\limsup_{R\to\infty}\left|\frac{2}{R^2}\int_0^R r\,dN_1(r) - \frac{k_1}{\pi}\right| \leq \limsup_{R\to\infty}\frac{1}{\alpha}\int_R^{R(1+\alpha)}\left(\frac{1}{r} - \frac{r}{R^2(1+\alpha)^2}\right)dN_1(r)$$
$$+ \frac{1}{\alpha}\left|1 - 2\alpha - \frac{1}{(1+\alpha)^2}\right|\limsup_{R\to\infty}\int_0^R \frac{r}{R^2}dN_1(r) + \frac{k_1}{\pi}\left|\frac{\log(1+\alpha)-\alpha}{\alpha}\right|.$$

Replacing the terms on the right above by larger terms gives

$$\limsup_{R\to\infty}\left|\frac{2}{R^2}\int_0^R r\,dN_1(r) - \frac{k_1}{\pi}\right| \leq \limsup_{R\to\infty}\frac{1}{\alpha}\left(\frac{1}{R} - \frac{R}{R^2(1+\alpha)^2}\right)\int_R^{R(1+\alpha)}dN_1(r)$$
$$(12.03)\qquad + \frac{1}{\alpha}\left|1 - 2\alpha - \frac{1}{(1+\alpha)^2}\right|\limsup_{R\to\infty}\frac{1}{R}\int_0^R dN_1(r) + \frac{k_1}{\pi}\left|\frac{\log(1+\alpha)-\alpha}{\alpha}\right|.$$

Since the zeros in the right half-plane are contained among all the zeros of $F(z)$,

$$\limsup_{R\to\infty} \frac{1}{R}\int_R^{R(1+\alpha)} dN_1(r) \leqq \limsup_{R\to\infty} \frac{1}{R}\int_R^{R(1+\alpha)} dN(r) = \lim_{R\to\infty} \frac{N(R+R\alpha)-N(R)}{R}.$$

And by (11.06) this becomes

$$\limsup_{R\to\infty} \frac{1}{R}\int_R^{R(1+\alpha)} dN_1(r) \leqq \frac{2k_1\alpha}{\pi}.$$

Also

$$\limsup_{R\to\infty} \frac{1}{R}\int_0^R dN_1(r) = \limsup_{R\to\infty} \frac{N_1(R)}{R} \leqq \limsup_{R\to\infty} \frac{N(R)}{R} = \frac{2k_1}{\pi}.$$

Moreover for small α

$$\frac{1}{\alpha}\left(1 - 2\alpha - \frac{1}{(1+\alpha)^2}\right) = O(\alpha), \qquad \frac{\log(1+\alpha)-\alpha}{\alpha} = O(\alpha).$$

Using these results, (12.03) becomes

$$\limsup_{R\to\infty} \left| \frac{2}{R^2}\int_0^R r\,dN_1(r) - \frac{k_1}{\pi} \right| = O(\alpha).$$

But α can be chosen arbitrarily small. Thus

$$\lim_{R\to\infty} \frac{2}{R^2}\int_0^R r\,dN_1(r) = \frac{k_1}{\pi}.$$

This can be written as

$$\int_0^R r\,dN_1(r) = R^2\left(\frac{k_1}{2\pi} + o(1)\right).$$

Integrating by parts gives

$$RN_1(R) - \int_0^R N_1(r)\,dr = R^2\left(\frac{k_1}{2\pi} + o(1)\right). \tag{12.04}$$

If

$$xh(x) - \int_0^x h(u)\,du = x^2 g(x), \tag{12.05}$$

then we shall prove that

$$h(x) = xg(x) + \int_0^x g(u)\,du + C_0 \tag{12.06}$$

where C_0 is a constant. For let

$$h(x) = xg(x) + \int_0^x g(u)\,du + t(x).$$

From (12.05), $h(x) - xg(x)$ is continuous and therefore $t(x)$ is continuous. Putting $h(x)$ into (12.05), we have

$$x^2 g(x) + x \int_0^x g(u)\, du + xt(x) - \int_0^x ug(u)$$
$$- \int_0^x du \int_0^u g(y)\, dy - \int_0^x t(u)\, du = x^2 g(x).$$

Integrating $\int_0^x ug(u)\, du$ by parts the above equation becomes

$$xt(x) - \int_0^x t(u)\, du = 0.$$

Since $t(x)$ is continuous, this equation implies that $t(x)$ is differentiable. Hence differentiating,

$$x \frac{dt(x)}{dx} = 0.$$

That is, $t(x)$ is a constant. This proves (12.06).

If we now apply (12.06) to (12.04) we get

$$N_1(R) = \frac{k_1}{\pi} R + o(R).$$

The same proof holds for $N_2(r)$, and this completes the proof of the lemma.

Proof of Theorem XVII. Since the zeros of $F(z)$ are $r_n/\cos\theta_n$ and are therefore larger in modulus than those of $f(z)$, it follows that

$$n_1(r) \geqq N_1(r).$$

Thus by (12.01)

$$\liminf_{r\to\infty} \frac{n_1(r)}{r} \geqq \frac{k_1}{\pi}. \tag{12.07}$$

Let $n_3(r)$ be the number of zeros, z_n, of $f(z)$ less than r in modulus in the right half-plane for which $|\operatorname{am} z_n| \geqq \delta$ where δ is a small positive quantity. Since by Lemma 9.1

$$\sum_1^\infty \frac{|\sin\theta_n|}{r_n} < \infty,$$

it follows that

$$\int_0^\infty \frac{dn_3(r)}{r} \sin\delta < \infty.$$

Integrating by parts

$$\int_0^\infty \frac{n_3(r)}{r^2}\, dr < \infty. \tag{12.08}$$

Since $n_3(r)$ is an increasing function of r,

$$\frac{n_3(R)}{R} = n_3(R) \int_R^\infty \frac{dr}{r^2} \leqq \int_R^\infty \frac{n_3(r)}{r^2}\, dr.$$

Using (12.08) this gives

$$\lim_{R\to\infty} \frac{n_3(R)}{R} = 0. \tag{12.09}$$

Let us now consider the zeros z_n of $f(z)$ in the right half-plane for which $|\operatorname{am} z_n| < \delta$. Then the number of these zeros less than r in modulus is $n_1(r) - n_3(r)$. Since the zeros of $F(z)$ are in the form $r_n/\cos\theta_n$, it follows that

$$n_1(r) - n_3(r) \leqq N_1\left(\frac{r}{\cos\delta}\right).$$

Thus

$$\limsup_{r\to\infty} \frac{n_1(r) - n_3(r)}{r} \leqq \limsup_{r\to\infty} \frac{N_1(r/\cos\delta)}{r}.$$

By (12.01) and (12.09) this gives

$$\limsup_{r\to\infty} \frac{n_1(r)}{r} \leqq \frac{k_1}{\pi\cos\delta}.$$

Since δ can be taken arbitrarily close to zero, this inequality and (12.07) give

$$\lim_{r\to\infty} \frac{n_1(r)}{r} = \frac{k_1}{\pi}.$$

If we set $k_1/\pi = B$ and recall that $k_1 \leqq k$, then

$$\lim_{r\to\infty} \frac{n_1(r)}{r} = B \leqq \frac{k}{\pi}.$$

The same proof applies to $n_2(r)$, and this completes the proof of Theorem XVII.

13. A related theorem. In this section we shall prove Theorem XI.

Proof of Theorem XI. As in the proof of Theorem XVII we introduce $F(z)$ with real zeros. We recall that $|F(x)| \leqq |f(x)|$ and thus by (6.05)

$$F(\lambda_n) = O(e^{-\delta\lambda_n}). \tag{13.01}$$

We also recall that

$$\lim_{r\to\infty} \frac{N_1(r)}{r} = B \leqq \frac{k}{\pi}. \tag{13.02}$$

Since the Pólya maximum density of $\{\lambda_n\}$ is $D > k/\pi$, it follows that $D > B$. Let $D - B = \alpha$. Then $\alpha > 0$.

From the definition of Pólya maximum density it follows that there exists a value of ξ, $\xi_0 < 1$, such that

$$\limsup_{r\to\infty} \frac{\Lambda(r) - \Lambda(r\xi_0)}{r - r\xi_0} \geqq D - \tfrac{1}{8}\,\alpha.$$

From this it further follows that there exists a sequence of values $\{r_m\}$, $(m > 0)$, such that $r_m \to \infty$ and

$$\frac{\Lambda(r_m) - \Lambda(r_m\xi_0)}{r_m - r_m\xi_0} \geqq D - \tfrac{1}{4}\,\alpha.$$

On the other hand, by (13.02) it follows that for sufficiently large m

$$\frac{N_1(r_m) - N_1(r_m\xi_0)}{r_m - r_m\xi_0} \leqq B + \tfrac{1}{4}\,\alpha.$$

In other words, the number of zeros of $F(x)$ in the interval $(r_m\xi_0\,,\, r_m)$ is at most

$$(B + \tfrac{1}{4}\alpha)r_m(1 - \xi_0) + 2,$$

whereas the number of $\{\lambda_n\}$ in this interval is at least

$$(D - \tfrac{1}{4}\alpha)r_m(1 - \xi_0) - 2.$$

We recall that $\lambda_{n+1} - \lambda_n \geqq c > 0$. Associate with each λ_n the interval $(\lambda_n - \frac{1}{8}c, \lambda_n + \frac{1}{8}c)$. These intervals are separated from each other. Since the number of λ_n exceeds the number of zeros of $F(z)$ in $(r_m\xi_0\,,\, r_m)$ by at least

$$(D - \tfrac{1}{4}\alpha)r_m(1 - \xi_0) - (B + \tfrac{1}{4}\alpha)r_m(1 - \xi_0) - 4 = \tfrac{1}{2}\alpha r_m(1 - \xi_0) - 4,$$

the number of intervals $(\lambda_n - \frac{1}{8}c, \lambda_n + \frac{1}{8}c)$ in $(r_m\xi_0\,,\, r_m)$ which contain no zeros of $F(z)$ is at least

$$\tfrac{1}{2}\alpha r_m(1 - \xi_0) - 6. \tag{13.03}$$

But since $F(z)$ has only real zeros and since $F(z) = O(e^{A|z|})$, it follows from a theorem of Laguerre[2] that $F'(x)$ vanishes once and only once between successive zeros of $F(x)$. We recall that $F(x)$ is real valued. Thus between each adjacent pair of zeros of $F(x)$, $|F(x)|$ increases steadily from zero to its maximum value and then decreases steadily to zero. Since each interval $(\lambda_n - \frac{1}{8}c, \lambda_n + \frac{1}{8}c)$ which we are considering contains no zero of $F(x)$, it now follows that in one of the two intervals $(\lambda_n\,,\, \lambda_n + \frac{1}{8}c)$ and $(\lambda_n - \frac{1}{8}c, \lambda_n)$, $|F(x)| \leqq |F(\lambda_n)|$. Thus

$$\int_{\lambda_n - c/8}^{\lambda_n + c/8} \log^- |F(x)|\,dx \leqq \tfrac{1}{8}\,c \log^- |F(\lambda_n)|. \tag{13.04}$$

By (13.01), this gives

$$\int_{\lambda_n - c/8}^{\lambda_n + c/8} \log^- |F(x)|\,dx \leqq -\tfrac{1}{8}\,c\delta\lambda_n + A, \tag{13.05}$$

where A is a constant independent of λ_n.

[2] Titchmarsh, *Theory of Functions*, Oxford, 1932, p. 266.

By (13.03) there are at least $\frac{1}{2}\alpha r_m(1 - \xi_0) - 6$ intervals in $(r_m\xi_0, r_m)$ where (13.05) holds. Moreover in $(r_m\xi_0, r_m)$, $\lambda_n \geqq r_m\xi_0$. Thus

$$\int_{r_m\xi_0}^{r_m} \log^- |F(x)| \, dx \leqq \tfrac{1}{2}\alpha r_m(1 - \xi_0)(-\tfrac{1}{8}c\delta r_m\xi_0) + Ar_m,$$

or

$$\int_{r_m\xi_0}^{r_m} \log \; |F(x)| \frac{dx}{x^2} \leqq -\frac{1}{16}\alpha c\delta\xi_0(1 - \xi_0) + \frac{A}{\xi_0^2 r_m}.$$

Letting $m \to \infty$,

$$\limsup_{m\to\infty} \int_{r_m\xi_0}^{r_m} \log^- |F(x)| \frac{dx}{x^2} < 0. \tag{13.06}$$

Since $r_m \to \infty$, (13.06) implies that

$$\int_1^\infty \log^- |F(x)| \frac{dx}{x^2} = -\infty. \tag{13.07}$$

But $|F(x)| \leqq |f(x)|$ and (9.05) imply that

$$\int_{-\infty}^\infty \frac{\log^+ |F(x)|}{1 + x^2} dx < \infty. \tag{13.08}$$

By Carleman's theorem used in exactly the same way as in getting (9.06), (13.08) implies that

$$\int_{-\infty}^\infty \frac{\log^- |F(x)|}{1 + x^2} dx > -\infty$$

if $F(z)$ is not identically zero. But this contradicts (13.07). Thus $F(z)$, and therefore also $f(z)$, is identically zero. This completes the proof of Theorem XI.

14. Another related theorem. *Proof of Theorem* XIV. The proof of Theorem XIV proceeds in much the same way as that of Theorem XI. In Theorem XIV we also introduce $F(z)$, and, since $|F(x)| \leqq |f(x)|$,

$$F(\lambda_n) = O(e^{-\theta(\lambda_n)}), \tag{14.01}$$

for a sequence of λ_n having an ordinary density D. Also

$$\lim_{r\to\infty} \frac{N_1(r)}{r} = B \leqq \frac{k}{\pi}.$$

Since $D > k/\pi$, $D > B$. Thus, as before, if $\alpha = D - B$ then $\alpha > 0$. Since λ_n has an ordinary density D,

$$\lim_{r\to\infty} \frac{\Lambda(2r) - \Lambda(r)}{r} = D.$$

Setting $r = 2^m$, it follows that for sufficiently large m

$$\frac{\Lambda(2^{m+1}) - \Lambda(2^m)}{2^m} \geqq D - \tfrac{1}{4}\alpha.$$

On the other hand, by (14.02) for sufficiently large m

$$\frac{N(2^{m+1}) - N(2^m)}{2^m} \leqq B + \tfrac{1}{4}\alpha.$$

Thus the number of zeros of $F(x)$ in the interval $(2^m, 2^{m+1})$ is at most

$$(B + \tfrac{1}{4}\alpha)2^m + 2,$$

whereas the number of $\{\lambda_n\}$ in this interval is at least

$$(D - \tfrac{1}{4}\alpha)2^m - 2.$$

As in §13 there must therefore be at least $\frac{1}{2}\alpha 2^m - 6$ intervals $(\lambda_n - \frac{1}{8}c, \lambda_n + \frac{1}{8}c)$ in $(2^m, 2^{m+1})$ which contain no zeros of $F(x)$. Thus, as before, by Laguerre's theorem for each of these intervals $(\lambda_n - \frac{1}{8}c, \lambda_n + \frac{1}{8}c)$ (13.04) holds. That is,

$$\int_{\lambda_n - c/8}^{\lambda_n + c/8} \log^- |F(x)|\, dx \leqq \tfrac{1}{8}c \log^- |F(\lambda_n)|.$$

By (14.01) this becomes

$$\text{(14.02)} \qquad \int_{\lambda_n - c/8}^{\lambda_n + c/8} \log^- |F(x)|\, dx \leqq -\tfrac{1}{8}c\theta(\lambda_n) + A.$$

Since in $(2^m, 2^{m+1})$ there are at least $\frac{1}{2}\alpha 2^m - 6 > \frac{1}{4}\alpha 2^m$ such intervals and since, in $(2^m, 2^{m+1})$, $\lambda_n \geqq 2^m$, we have from (14.02)

$$\begin{aligned}\int_{2^m}^{2^{m+1}} \log^- |F(x)|\, dx &\leqq (\tfrac{1}{4}\alpha 2^m)(-\tfrac{1}{8}c\theta(2^m)) + \alpha A 2^{m+1} \\ &\leqq -\alpha c 2^{m-5}\theta(2^m) + \alpha A 2^{m+1}.\end{aligned}$$

From this

$$\begin{aligned}\int_{2^m}^{2^{m+1}} \log^- |F(x)| \frac{dx}{x^2} &\leqq -\frac{\alpha c}{128}\frac{\theta(2^m)}{2^m} + \frac{2\alpha A}{2^m} \\ &\leqq -\frac{\alpha c}{256}\frac{\theta(2^m)}{2^m}, \qquad m \to \infty.\end{aligned}$$

Since $\theta(x)$ is monotone increasing, it follows that

$$\int_{2^m}^{2^{m+1}} \log^- |F(x)| \frac{dx}{x^2} \leqq -\frac{\alpha c}{1000}\int_{2^{m-1}}^{2^m} \frac{\theta(x)}{x^2}\, dx$$

for large m. Adding the above inequalities and recalling that

$$\int_1^\infty \frac{\theta(x)}{x^2}\, dx = \infty,$$

it follows that

$$\int_1^\infty \log^- |F(x)| \frac{dx}{x^2} = -\infty.$$

But as in the proof of Theorem XI, this implies that $F(z)$ is identically zero. This means $f(z)$ must be identically zero and completes the proof of Theorem XIV.

CHAPTER IV

ON NON-HARMONIC FOURIER SERIES[1]

15. Proof of an underlying inequality. In this chapter we consider the question of expanding a function $f(x)$ into a series of trigonometric functions, $\{e^{i\lambda_n x}\}$. Up to this point whatever theorems we have proved involving $\{e^{i\lambda_n x}\}$ have been essentially closure theorems. That is, we have shown under various conditions on λ_n that if $f(x) \epsilon L^p(-\pi, \pi)$ and if

$$\int_{-\pi}^{\pi} f(x)e^{i\lambda_n x}\, dx = 0,$$

then $f(x)$ is a null function. In general such results do not imply that $f(x)$ can be represented by a series $\sum a_n e^{i\lambda_n x}$. Even in the case where $f(x) \epsilon L^2(-\pi, \pi)$, all that is implied by closure is that given any ϵ it is possible to find a polynomial in $\{e^{i\lambda_n x}\}$, $P_\epsilon(x)$, such that

$$\int_{-\pi}^{\pi} |f(x) - P_\epsilon(x)|^2\, dx < \epsilon.$$

Therefore it is of considerable interest to find conditions under which it is possible to get a series representation for $f(x)$ in terms of $\{e^{i\lambda_n x}\}$ analogous to the Fourier series. Such series were studied by Paley and Wiener[2] who called them non-harmonic Fourier series.

Paley and Wiener proved that *if*

$$|\lambda_n - n| \leqq D < \frac{1}{\pi^2}, \qquad -\infty < n < \infty, \tag{15.01}$$

then the sequence $\{e^{i\lambda_n x}\}$ is closed in $L^2(-\pi, \pi)$ and possesses a unique biorthogonal set $\{h_n(x)\}$, such that the series

$$\sum_{-\infty}^{\infty} \left\{\frac{e^{inx}}{2\pi} \int_{-\pi}^{\pi} f(\xi)e^{-in\xi}\, d\xi - e^{i\lambda_n x} \int_{-\pi}^{\pi} f(\xi)h_n(\xi)\, d\xi\right\}$$

converges uniformly to zero over any interval $(-\pi + \delta \leqq x \leqq \pi - \delta)$ for any positive δ, and over any such interval the summability properties of

$$\sum_{-\infty}^{\infty} e^{i\lambda_n x} \int_{-\pi}^{\pi} f(\xi)h_n(\xi)\, d\xi \tag{15.02}$$

are uniformly the same as those of the Fourier series of $f(x)$.

[1] Cf. N. Levinson, *On non-harmonic Fourier series*, Annals of Mathematics, vol. 37 (1936), p. 919.

[2] Paley and Wiener, loc. cit., chap. 7, p. 108.

In other words, if (15.01) holds, the non-harmonic Fourier series (15.02) converges in exactly the same way as the ordinary Fourier series.

Paley and Wiener asked two questions about their theorem. First is it necessary that $D < 1/\pi^2$ in (15.01) and second is there a theorem if $f(x) \in L(-\pi, \pi)$? The answer to the first question is that if $f(x) \in L^2(-\pi, \pi)$ it suffices for $D < \frac{1}{4}$, and this is a best possible result. The answer to the second question is that there is no general theory if $f(x) \in L(-\pi, \pi)$ but there is a general theory if $f(x) \in L^p(-\pi, \pi)$, $1 < p \leqq 2$. These results are contained in the following theorems.

THEOREM XVIII. *If $\{\lambda_n\}$ is a sequence and D a constant such that*

$$(15.03)\qquad |\lambda_n - n| \leqq D < \frac{p-1}{2p}, \qquad -\infty < n < \infty,$$

for some p, $1 < p \leqq 2$, then the set $\{e^{i\lambda_n t}\}$ is closed $L^p(-\pi, \pi)$ and possesses a unique biorthogonal set $\{h_n(x)\}$ such that for any $f(x) \in L^p(-\pi, \pi)$ the series

$$(15.04)\qquad \sum_{-\infty}^{\infty} \left\{\frac{e^{inx}}{2\pi} \int_{-\pi}^{\pi} f(\xi)e^{-in\xi}\, d\xi - e^{i\lambda_n x} \int_{-\pi}^{\pi} f(\xi)h_n(\xi)\, d\xi\right\}$$

converges uniformly to zero over any interval $(-\pi + \delta \leqq x \leqq \pi - \delta)$ for any $\delta > 0$. Moreover the difference of the weighted sums (Riesz, Abel, and so on) of the non-harmonic and ordinary Fourier series also converges uniformly to zero over $(-\pi + \delta \leqq x \leqq \pi - \delta)$.

In other words, the convergence and summability properties of the series (15.04) are exactly the same for the ordinary Fourier series for $f(x)$ over $-\pi < x < \pi$.

THEOREM XIX. *If (15.03) is replaced by*

$$(15.05)\qquad |\lambda_n - n| \leqq \frac{p-1}{2p},$$

then the results of Theorem XVIII no longer hold.

Combining these two theorems, it is clear that there exists no general L theory of non-harmonic Fourier series since as $p \to 1$, $D \to 0$.

The question as to whether the partial sums of the non-harmonic Fourier series of $f(x)$, where $f(x) \in L^p(-\pi, \pi)$, converge in the pth mean to $f(x)$ is still open. In the case of the result of Paley and Wiener, loc. cit., with $D < 1/\pi^2$ and $f(x) \in L^2(-\pi, \pi)$, it was shown that the partial sums converged in the mean to $f(x)$ over $(-\pi, \pi)$. Naturally these results are of interest only at the end-points of the interval $(-\pi, \pi)$ since in the interior the equi-convergence with the ordinary Fourier series assures such convergence in the mean. Thus for $\epsilon > 0$

$$\lim_{N\to\infty} \int_{-\pi+\epsilon}^{\pi-\epsilon} \left|f(x) - \sum_{-N}^{N} e^{i\lambda_n x} \int_{-\pi}^{\pi} f(\xi)h_n(\xi)\, d\xi\right|^p dx = 0.$$

The problem of convergence in the mean therefore reduces to showing that, as $\epsilon \to +0$,

$$\limsup_{N\to\infty}\left(\int_{-\pi}^{-\pi+\epsilon}+\int_{\pi-\epsilon}^{\pi}\right)\left|\sum_{-N}^{N} e^{i\lambda_n x}\int_{-\pi}^{\pi} f(\xi)h_n(\xi)\,d\xi\right|^p dx = o(1).$$

In proving Theorem XVIII the following result is of basic importance.

THEOREM XX. *Let* $\{\lambda_n\}$ *satisfy* (15.03) *and let*

$$F(x) = \prod_1^\infty\left(1-\frac{x}{\lambda_n}\right)\left(1-\frac{x}{\lambda_{-n}}\right). \tag{15.06}$$

Then there exists an absolute constant A *such that*

$$\int_{-\infty}^{\infty} |F(x)|^p\,dx < \frac{A}{(p-1-2pD)^2}, \tag{15.07}$$

and

$$\int_c^{\infty} |F(x)|^p\,dx < \frac{Ac^{-(p-1-2pD)}}{(p-1-2pD)^2} \tag{15.08}$$

for $c > 1$.

As we shall see, to prove that Theorem XX is a best possible one is quite simple.

First we shall prove Theorem XX It is convenient to use $A_k > 0$, $k = 0, 1, \cdots$, for absolute constants.

LEMMA 15.1. *If* $F(x)$ *is defined as in Theorem* XX, *then*

$$\int_N^{2N} |F(x)|^p\,dx \leqq A_0 N^{-p+4pD}\int_N^{2N}\frac{|P(x)|^{2pD}}{(2N+1-x)^{2pD}}\,dx \tag{15.09}$$

where

$$P(x) = \prod_N^{2N}\left(\frac{\alpha_n+n-D-x-2i}{n-D-x-2i}\right) \tag{15.10}$$

and α_n *are constants such that* $0 \leqq \alpha_n \leqq 1$.

Proof of Lemma 15.1. Let N be some positive integer. If $N < x < 2N$ it is clear from $|\lambda_n - n| \leqq D$ that

$$\begin{aligned} 1+\frac{x}{n-D} &\geqq \left|1-\frac{x}{\lambda_{-n}}\right|, && 0<n<\infty,\\ \frac{x}{n-D}-1 &\geqq \left|\frac{x}{\lambda_n}-1\right|, && 0<n<N,\\ 1-\frac{x}{n+D} &\geqq \left|1-\frac{x}{\lambda_n}\right|, && 2N<n<\infty. \end{aligned} \tag{15.11}$$

Therefore, for $N < x < 2N$

$$|F(x)| \leqq \prod_1^{2N} \left|1 - \frac{x}{n-D}\right|\left|1 + \frac{x}{n-D}\right| \cdot \prod_{2N+1}^{\infty} \left|1 - \frac{x}{n+D}\right|\left|1 + \frac{x}{n-D}\right| \prod_N^{2N} \left|\frac{1 - x/\lambda_n}{1 - x/(n-D)}\right|.$$

And since $n - D \leqq \lambda_n$ for $n > 0$, it follows from this that

$$|F(x)| \leqq \prod_1^{2N} \left|1 - \frac{x}{n-D}\right|\left|1 + \frac{x}{n-D}\right| \cdot \prod_{2N+1}^{\infty} \left|1 - \frac{x}{n+D}\right|\left|1 + \frac{x}{n-D}\right| \prod_N^{2N} \left|\frac{\lambda_n - x}{n - D - x}\right|. \tag{15.12}$$

Let us recall the following well-known properties of the gamma function:

$$\Gamma(x+1) = x\Gamma(x), \tag{15.13}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \tag{15.14}$$

$$\Gamma(x+1) \sim (2\pi)^{1/2} x^{x+1/2} e^{-x}. \tag{15.15}$$

Using (15.13)

$$\prod_K^M \left|1 - \frac{x}{n-a}\right|\left|1 + \frac{x}{n-b}\right| = \left|\frac{\Gamma(M+1-a-x)\Gamma(M+1-b+x)\Gamma(K-a)\Gamma(K-b)}{\Gamma(K-a-x)\Gamma(K-b+x)\Gamma(M+1-a)\Gamma(M+1-b)}\right| \tag{15.16}$$

Let $K = 2N + 1$, $a = -D$, $b = D$ and $M \to \infty$. Then

$$\prod_{2N+1}^{\infty} \left|1 - \frac{x}{n+D}\right|\left|1 + \frac{x}{n-D}\right| = \left|\frac{\Gamma(2N+1+D)\Gamma(2N+1-D)}{\Gamma(2N+1+D-x)\Gamma(2N+1-D+x)}\right| \cdot \lim_{M\to\infty} \left|\frac{\Gamma(M+1+D-x)\Gamma(M+1-D+x)}{\Gamma(M+1+D)\Gamma(M+1-D)}\right|. \tag{15.17}$$

Using (15.15), we have

$$\begin{aligned}
\lim_{M\to\infty} &\left|\frac{\Gamma(M+1+D-x)\Gamma(M+1-D+x)}{\Gamma(M+1+D)\Gamma(M+1-D)}\right| \\
&= \lim_{M\to\infty} \left\{\left(\frac{M+D-x}{M+D}\right)^{M+D+1/2} \left(\frac{M-D-x}{M-D}\right)^{M-D+1/2} \left(\frac{M-D+x}{M+D-x}\right)^{x}\right\} \\
&= \lim_{M\to\infty} \left\{\left(1 - \frac{x}{M+D}\right)^{M+D} \left(1 + \frac{x}{M-D}\right)^{M-D}\right\} \\
&= e^{-x}e^{x} = 1.
\end{aligned}$$

Thus (15.17) becomes

$$\prod_{2N+1}^{\infty}\left|1-\frac{x}{n+D}\right|\left|1+\frac{x}{n-D}\right| = \left|\frac{\Gamma(2N+1+D)\Gamma(2N+1-D)}{\Gamma(2N+1+D-x)\Gamma(2N+1-D+x)}\right|. \tag{15.18}$$

Setting $K = 1$, $M = 2N$, and $a = b = D$ in (15.16) gives

$$\prod_{1}^{2N}\left|1-\frac{x}{n-D}\right|\left|1+\frac{x}{n-D}\right| = \left|\frac{\Gamma^2(1-D)\Gamma(2N+1-D-x)\Gamma(2N+1-D+x)}{\Gamma(1-D+x)\Gamma(1-D-x)\Gamma^2(2N+1-D)}\right|.$$

By (15.14) this becomes

$$\prod_{1}^{2N}\left|1-\frac{x}{n-D}\right|\left|1+\frac{x}{n-D}\right| = \left|\frac{\Gamma^2(1-D)\sin\pi(x+D)\Gamma(x+D)\Gamma(2N+1-D-x)\Gamma(2N+1-D+x)}{\pi\Gamma(1-D+x)\Gamma^2(2N+1-D)}\right|.$$

Using this and (15.18), (15.12) becomes

$$|F(x)| \leqq \frac{\Gamma^2(1-D)\sin\pi(x+D)}{\pi}\left(\frac{\Gamma(x+D)}{\Gamma(x-D+1)}\right)\left(\frac{\Gamma(2N+1+D)}{\Gamma(2N+1-D)}\right) \cdot\left(\frac{\Gamma(2N+1-D-x)}{\Gamma(2N+1+D-x)}\right)\prod_{N}^{2N}\left|\frac{\lambda_n-x}{n-D-x}\right| \tag{15.19}$$

for $N < x < 2N$.

By (15.15) for large M and fixed a and b

$$\begin{aligned}\frac{\Gamma(M+a)}{\Gamma(M+b)} &< 2e^{b-a}\left(\frac{M-1+a}{M-1+b}\right)^{M-1/2+b}(M-1+a)^{a-b}\\ &< 5e^{b-a}\left(1+\frac{a-b}{M-1+b}\right)^{M-1+b}M^{a-b}\\ &< 10e^{b-a}e^{a-b}M^{a-b} = 10M^{a-b}.\end{aligned}$$

Using this on each of the terms in parenthesis on the right of (15.19), and recalling that $N < x < 2N$, gives

$$|F(x)| \leqq A_1N^{-1+4D}\frac{|\sin\pi(x+D)|}{(2N+1-x)^{2D}}\prod_{N}^{2N}\left|\frac{\lambda_n-x}{n-D-x}\right|. \tag{15.20}$$

When $|n - D - x| < \frac{1}{2}$ we have

$$\left|\frac{\lambda_n-x}{n-D-x}\right||\sin\pi(x+D)| \leqq \pi. \tag{15.21}$$

When $\frac{1}{2} \leqq |n - D - x|$ the inequality

$$1 + t \leqq e^t, \qquad -\infty < t < \infty,$$

gives

$$(15.22)\qquad \left|\frac{\lambda_n - x}{n - D - x}\right| = 1 + \frac{\lambda_n - n + D}{n - D - x} \leqq \exp\frac{\lambda_n - n + D}{n - D - x}.$$

Clearly

$$\left|\frac{4D}{(n - D - x)(n - D - x - 2i)}\right| \leqq \frac{4D}{(n - D - x)^2}$$

Since $|\lambda_n - n + D| \leqq 2D$,

$$\frac{4D}{(n - D - x)^2} \geqq \left|\frac{(\lambda_n - n + D)2i}{(n - D - x)(n - D - x - 2i)}\right|$$
$$= \left|\frac{\lambda_n - n + D}{n - D - x} - \frac{\lambda_n - n + D}{n - D - x - 2i}\right|$$
$$\geq \Re\left\{\frac{\lambda_n - n + D}{n - D - x} - \frac{\lambda_n - n + D}{n - D - x - 2i}\right\}.$$

Or

$$\frac{\lambda_n - n + D}{n - D - x} \leqq \frac{4D}{(n - D - x)^2} + \Re\frac{\lambda_n - n + D}{n - D - x - 2i}.$$

Using this in (15.22),

$$\left|\frac{\lambda_n - x}{n - D - x}\right| \leqq \left|\exp\left(\frac{\lambda_n - n + D}{n - D - x - 2i} + \frac{4D}{(n - D - x)^2}\right)\right|$$

for $\frac{1}{2} \leqq |n - D - x|$. Combining this with (15.21),

$$\prod_N^{2N}\left|\frac{\lambda_n - x}{n - D - x}\right| |\sin \pi(x + D)|$$
$$\leqq \pi\left|\exp\left(\sum_N^{2N}\frac{\lambda_n - n + D}{n - D - x - 2i} + 8D\sum_1^{\infty}\frac{1}{(n - \frac{1}{2})^2} + 2D\right)\right|$$
$$< A_2\left|\exp\sum_N^{2N}\frac{\lambda_n - n + D}{n - D - x - 2i}\right|.$$

If we set

$$\alpha_n = \frac{\lambda_n - n + D}{2D},$$

then clearly $0 \leqq \alpha_n \leqq 1$ and the above inequality becomes

$$(15.23)\qquad \prod_N^{2N}\left|\frac{\lambda_n - x}{n - D - x}\right| |\sin \pi(x + D)| < A_2 \prod_N^{2N}\left|\exp\left\{\frac{\alpha_n}{n - D - x - 2i}\right\}\right|^{2D}.$$

From the power series expansion for e^z,

$$|e^z| \leqq |1 + z| + |z|^2, \qquad |z| \leqq \tfrac{1}{2}. \tag{15.24}$$

But for $|z| \leqq \frac{1}{2}$, $|1 + z| \geqq \frac{1}{2}$. Thus

$$|1 + z| + |z|^2 \leqq |1 + z|(1 + 2|z|^2),$$

and (15.24) becomes

$$|e^z| \leqq |1 + z|(1 + 2|z|^2), \qquad |z| \leqq \tfrac{1}{2}.$$

Using this in (15.23) gives

$$\prod_N^{2N} \left|\frac{\lambda_n - x}{n - D - x}\right| |\sin \pi(x + D)| < A_3 \prod_N^{2N} \left|\frac{\alpha_n + n - D - x - 2i}{n - D - x - 2i}\right|^{2D}.$$

This inequality in (15.20) gives

$$|F(x)| \leqq A_4 \frac{N^{-1+4D}}{(2N + 1 - x)^{2D}} \prod_N^{2N} \left|\frac{\alpha_n + n - D - x - 2i}{n - D - x - 2i}\right|^{2D} \tag{15.25}$$

for $N < x < 2N$. Taking the pth power and integrating, we have (15.09). This completes the proof of this lemma.

There is no simple majorant for $F(x)$ that is sufficiently close to $F(x)$ in magnitude to give a satisfactory appraisal of its size. The reason for this is that, depending on the $\{\lambda_n\}$, $F(x)$ can be large at some places but if so must compensate by being small in other places. Thus a satisfactory majorant involves $\{\lambda_n\}$ or something depending on $\{\lambda_n\}$ such as $\{\alpha_n\}$. The relation (15.25) gives such a majorant.

LEMMA 15.2. *Let*

$$f(x) = \frac{(x - a_2 + ib) \cdots (x - a_{2k} + ib)}{(x - a_1 + ib) \cdots (x - a_{2k+1} + ib)} \tag{15.26}$$

where $0 \leqq a_1 < a_2 < \cdots < a_{2k+1} \leqq 1$ *and* $b \geqq 0$. *If* $0 \leqq t < 1$, *then*

$$\int_0^1 |f(x)|^t \, dx \leqq \frac{5}{1 - t}. \tag{15.27}$$

This inequality is of some interest in itself and will be investigated more closely at the end of the chapter.

Proof of Lemma 15.2. Let

$$H(z) = i\,\frac{(z - a_2) \cdots (z - a_{2k})}{(z - a_1) \cdots (z - a_{2k+1})}.$$

Using the partial fraction expansion for $H(z)$,

$$H(z) = i\left(\frac{c_1}{z - a_1} + \frac{c_3}{z - a_3} + \cdots + \frac{c_{2k+1}}{z - a_{2k+1}}\right).$$

It is clear that the sum of the c's is one. Since the a's form an increasing sequence, it is easily seen that $c_1 > 0$, $c_3 > 0$, $\cdots$. If we set $z = x + iy$,

$$\Re H(z) = \frac{c_1 y}{(x - a_1)^2 + y^2} + \cdots + \frac{c_{2k+1} y}{(x - a_{2k+1})^2 + y^2}.$$

Thus $\Re H(z) \geqq 0$ for $y \geqq 0$, and therefore

$$| \operatorname{am} H(z) | \leqq \tfrac{1}{2}\pi, \qquad y \geqq 0.$$

This result and $\Re H^t(z) = | H(z) |^t \cos (t \operatorname{am} H(z))$ give

$$| H(z) |^t \leqq \frac{\Re H^t(z)}{\cos \frac{1}{2}\pi t}.$$

But $\cos \frac{1}{2}\pi t = \sin \frac{1}{2}\pi(1 - t) > 1 - t$. Thus

$$(15.28) \qquad | H(z) |^t \leqq \frac{\Re H^t(z)}{1 - t}.$$

Since $H(z)$ has no zeros or poles in $y > 0$, $H^t(z)$ is analytic for $y > 0$ and its integral around any closed contour in that region is zero. Thus for any fixed ϵ, $1 > \epsilon > 0$,

$$\int_{-1}^{2} H^t(x + i\epsilon)\, dx + i \int_{\epsilon}^{1} H^t(2 + iy)\, dy - \int_{-1}^{2} H^t(x + i)\, dx - i \int_{\epsilon}^{1} H^t(-1 + iy)\, dy = 0.$$

Or

$$(15.29) \qquad \left| \int_{-1}^{2} H^t(x + i\epsilon)\, dx \right| \leqq \int_{0}^{1} | H(2 + iy) |^t\, dy + \int_{0}^{1} | H(-1 + iy) |^t\, dy + \int_{-1}^{2} | H(x + i) |^t\, dx.$$

Since the c's are all positive and their sum is 1, it follows from the partial fraction expansion of $H(z)$ that

$$(15.30) \qquad | H(z) | \leqq \frac{1}{\min_{0 \leq l \leq k} | z - a_{2l+1} |}.$$

Using this in (15.29),

$$\left| \int_{-1}^{2} H^t(x + i\epsilon)\, dx \right| \leqq 1 + 1 + 3 = 5.$$

Thus

$$\int_{-1}^{2} \Re H^t(x + i\epsilon)\, dx \leqq 5.$$

Using this in (15.28),

$$\int_{-1}^{2} |H(x + i\epsilon)|^t \, dx \leq \frac{5}{1 - t}, \tag{15.31}$$

for any $\epsilon > 0$, $0 < \epsilon < 1$. If $\epsilon \geq 1$, (15.31) is trivial by (15.30). If $\epsilon = 0$, (15.31) holds since it holds for all $\epsilon > 0$. Thus (15.31) is true for all $\epsilon \geq 0$ and (15.27) follows immediately.

Proof of Theorem XX. Clearly

$$\int_{N}^{2N} \frac{|P(x)|^{2pD}}{(2N + 1 - x)^{2pD}} \, dx \leq 10 \int_{N-1}^{2N+1} \frac{|P(x)|^{2pD}}{|2N + 1 - x - 2i|^{2pD}} \, dx. \tag{15.32}$$

If we recall the definition of $P(x)$ and set

$$y = \frac{x - N + 1}{N + 2},$$

then

$$\int_{N-1}^{2N+1} \frac{|P(x)|^{2pD}}{|2N + 1 - x - 2i|^{2pD}} \, dx$$
$$\leq 4N^{1-2pD} \int_0^1 \left| \frac{(y - a_2 + ib) \cdots (y - a_{2k} + ib)}{(y - a_1 + ib) \cdots (y - a_{2k+1} + ib)} \right|^{2pD} dy,$$

where $0 \leq a_1 < a_2 < \cdots < a_{2k+1} \leq 1$ and where $b = 2/(N + 2)$. By Lemma 15.2 it follows that

$$\int_{N-1}^{2N+1} \frac{|P(x)|^{2pD}}{|2N + 1 - x - 2i|^{2pD}} \, dx \leq \frac{20}{1 - 2pD} N^{1-2pD}.$$

Using this with (15.32) and (15.09),

$$\int_{N}^{2N} |F(x)|^p \, dx \leq \frac{A_5 N^{-(p-1-2pD)}}{1 - 2pD}.$$

If we set $N = 2^m, 2^{m+1}, \cdots$ and add, we have

$$\int_{2^m}^{\infty} |F(x)|^p \, dx \leq \frac{A_5 2^{-m(p-1-2pD)}}{(1 - 2pD)(1 - 2^{-(p-1-2pD)})} \leq \frac{10 A_5 2^{-m(p-1-2pD)}}{(p - 1 - 2pD)^2}.$$

Theorem XX follows immediately.

16. Proof of the main theorem. We shall first prove several lemmas concerning

$$G(w) = (w - \lambda_0) \prod_1^{\infty} \left(1 - \frac{w}{\lambda_n}\right)\left(1 - \frac{w}{\lambda_{-n}}\right) \tag{16.01}$$

where

$$|\lambda_n - n| \leq D < \frac{p - 1}{2p}. \tag{16.02}$$

We shall continue to use A_k as absolute positive constants.

Lemma 16.1. *If $w = u + iv$, then*

$$|G(w)| < A_6(|w| + 1)^{4D}e^{\pi|v|}, \tag{16.03}$$

$$|G(w)| > A_7|v|(|w| + 1)^{-1-4D}e^{\pi|v|}, \tag{16.04}$$

$$|G(\tfrac{1}{2} + iv)| > A_8. \tag{16.05}$$

Proof of Lemma 16.1. Let am $w = \theta$. If $u \geqq 0$, $|w| \geqq \frac{1}{2}$ and if N is determined by

$$N - \tfrac{1}{2} \leqq |w| \sec\theta < N + \tfrac{1}{2},$$

then

$$|w/\lambda_n| < \cos\theta, \qquad n > N.$$

Also, since $u \geqq 0$, w/λ_n lies in the right half-plane. In the accompanying diagram, Fig. 2, P represents w/λ_n and $OD = \cos\theta$. Clearly moving P closer

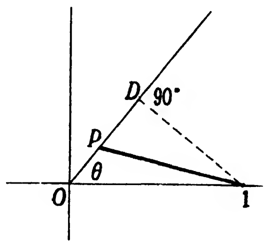

Fig 2

to D shortens the line from 1 to P. Thus

$$\left|1 - \frac{w}{\lambda_n}\right| \geqq \left|1 - \frac{w}{n - D}\right|, \qquad n > N.$$

Similarly

$$\left|\frac{w}{\lambda_n}\right| > \cos\theta, \qquad 0 < n < N,$$

and therefore

$$\left|1 - \frac{w}{\lambda_n}\right| \geqq \left|1 - \frac{w}{n + D}\right|, \qquad 0 < n < N.$$

Since $u \geqq 0$,

$$\left|1 - \frac{w}{\lambda_{-n}}\right| \geqq \left|1 + \frac{w}{n + D}\right|, \qquad n > 0.$$

Clearly

$$\left|1 - \frac{w}{\lambda_N}\right| \geqq \frac{|\lambda_N - w|}{N + D}.$$

Therefore

$$|G(w)| \geqq |w - D| \frac{|\lambda_N - w|}{N + D} \prod_1^{N-1} \left|1 - \frac{w}{n + D}\right| \left|1 + \frac{w}{n + D}\right|$$

$$\cdot \left|1 + \frac{w}{N + D}\right| \prod_{N+1}^{\infty} \left|1 - \frac{w}{n - D}\right| \left|1 + \frac{w}{n + D}\right|.$$

Using the gamma function in the same way as in Lemma 15.1, we have

$$(16.06)\quad \begin{aligned} |G(w)| &\geqq \frac{\Gamma^2(1 + D)}{\pi} |w - D| |\lambda_N - w| \left|\frac{\sin \pi(w - D)}{w - N - D}\right| \\ &\quad \cdot \left|\frac{\Gamma(w - D)\Gamma(N + 1 + D - w)\Gamma(N + 1 - D)}{\Gamma(w + 1 + D)\Gamma(N + 1 - D - w)\Gamma(N + 1 + D)}\right| \\ &\geqq A_9 \frac{|w| |\lambda_N - w| e^{\pi|v|}}{1 + |w - N|} N^{-2D} \left|\frac{\Gamma(w - D)\Gamma(N + 1 + D - w)}{\Gamma(w + 1 + D)\Gamma(N + 1 - D - w)}\right|. \end{aligned}$$

But, for $x \geqq 0$, using Stirling's formula, (41.07), and taking the real part,

$$(16.07)\quad \log |\Gamma(x + iy)| = (x - \tfrac{1}{2}) \log |x + iy| - y \tan^{-1} y/x + O(1)$$

as $|x + iy| \to \infty$. For $x \geqq 0$ and $2 > \alpha > 0$,

$$\left|y \tan^{-1} \frac{y}{x} - y \tan^{-1} \frac{y}{x + \alpha}\right| = \left|y \tan^{-1} \frac{\alpha y}{x(x + \alpha) + y^2}\right| < \left|y \tan^{-1} \frac{\alpha}{y}\right| < 10.$$

Thus using (16.07) in (16.06), and recalling that $|w| < (N + \frac{1}{2}) \cos \theta$,

$$(16.08)\quad \begin{aligned} |G(w)| &\geqq A_{10} \frac{|\lambda_N - w| e^{\pi|v|}}{N^{2D} |w|^{2D} (1 + |w - N|)^{1-2D}} \\ &\geqq A_{11} \frac{|\lambda_N - w| e^{\pi|v|}}{(1 + |w - N|)(1 + |w|)^{4D}}. \end{aligned}$$

Since $G(-w)$ is of the same form as $G(w)$, the results hold for all values of u and for $|w| \geqq \frac{1}{2}$. By precisely the same argument with $|w| < \frac{1}{2}$ and therefore with N now zero, it is clear that the result is true for all w. Inequality (16.04) is an immediate consequence of (16.08), and (16.05) is also an immediate consequence of (16.08). The proof of (16.03) proceeds in very much the same way as the proof of (16.08).

Lemma 16.2. *The functions* $\{h_n(x)\}$ *defined by*

$$(16.09)\quad h_n(x) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{2\pi} \int_{-A}^{A} \frac{G(u)}{(u - \lambda_n)G'(\lambda_n)} e^{-iux} du$$

form a sequence biorthogonal to $\{e^{i\lambda_n x}\}$ *over* $(-\pi, \pi)$, *and*

$$h_n(x) \in L^q(-\pi, \pi), \qquad q = \frac{p}{p-1}.$$

The limit in the mean in (16.09) *is of the qth order.*

Proof of Lemma 16.2. By Theorem XX

$$\frac{G(u)}{u - \lambda_n} \in L^p(-\infty, \infty).$$

Thus by Theorem G on Fourier transforms of functions in L^p, $h_n(x)$ defined as in (16.09) exists and belongs to $L^q(-\infty, \infty)$, $q = p/(p-1)$.

By (16.03)

$$\left| \frac{G(w)}{(w - \lambda_n)G'(\lambda_n)} \right| < \frac{A_{12} \mid \lambda_n \mid^{4D} e^{\pi|v|}}{(1 + \mid w \mid)^{1-4D} \mid G'(\lambda_n) \mid}.$$

Since $D < (p-1)/2p$ and $p \leqq 2$, $D < \frac{1}{4}$. Thus if $1 - 4D = \delta$, $\delta > 0$ and

$$\frac{G(w)}{(w - \lambda_n)G'(\lambda_n)} = O\left(\frac{e^{\pi|v|}}{\mid w \mid^{\delta}}\right)$$

as $\mid w \mid \to \infty$. Thus for $x < -\pi$ and $v \geqq 0$,

$$\frac{G(w)e^{-iwx}}{(w - \lambda_n)G'(\lambda_n)} = O\left(\frac{e^{(x+\pi)v}}{\mid w \mid^{\delta}}\right) = O\left(\frac{e^{-(|x|-\pi)v}}{\mid w \mid^{\delta}}\right).$$

Therefore

$$h_n(x) = \lim_{A \to \infty} \frac{1}{2\pi} \int_C \frac{G(w)e^{-iwx}}{(w - \lambda_n)G'(\lambda_n)}\, dw,$$

if this limit exists, where C is the closed contour formed by the line $(-A \leqq u \leqq A)$ and the semi-circular arc $(\mid w \mid = A, 0 \leqq \text{am}\, w \leqq \pi)$. But the integral of an analytic function around a closed contour is zero. Thus $h_n(x) = 0$, $x < -\pi$. A similar result holds if $x > \pi$, giving

$$h_n(x) = 0, \qquad \mid x \mid > \pi.$$

Again using the transform theorem, Theorem G,

$$\frac{G(u)}{(u - \lambda_n)G'(\lambda_n)} = \int_{-\pi}^{\pi} h_n(x)e^{iux}\, dx.$$

From this and the fact that $G(\lambda_m) = 0$ for all m it follows that

$$\int_{-\pi}^{\pi} h_n(x)e^{i\lambda_m x}\, dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

Thus $\{h_n(x)\}$ form a biorthogonal sequence to $\{e^{i\lambda_n x}\}$.

We can now prove Theorem XVIII.

Proof of Theorem XVIII. Let C denote a rectangular path in the complex ζ plane with vertices at $(N + \frac{1}{2} + iM, -N - \frac{1}{2} + iM, -N - \frac{1}{2} - iM, N + \frac{1}{2} - iM)$. Let $\phi(x) = 1, |x| < 1$, and let $\phi(x) = 0, |x| > 1$. The function $G(u)$ is defined as in (16.01).

Using residues and Lemma 16.2,

$$\begin{aligned}
&\operatorname*{l.i.m.}_{A\to\infty} \frac{1}{4\pi^2 i}\int_{-A}^{A} G(u)e^{-iuy}\,du \int_C \frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)}\,d\zeta \\
&\quad = \operatorname*{l.i.m.}_{A\to\infty} \frac{1}{2\pi}\int_{-A}^{A} G(u)e^{-iuy}\left[\sum_{-N}^{N}\frac{e^{i\lambda_n x}}{G'(\lambda_n)(u-\lambda_n)} - \frac{e^{iux}}{G(u)}\,\phi\left(\frac{u}{N+\frac{1}{2}}\right)\right]du \\
&\quad = \sum_{-N}^{N} h_n(y)e^{i\lambda_n x} - \frac{1}{2\pi}\int_{-N-1/2}^{N+1/2} e^{iu(x-y)}\,du \\
&\quad = \sum_{-N}^{N} h_n(y)e^{i\lambda_n x} - \frac{\sin(N+\frac{1}{2})(x-y)}{\pi(x-y)}.
\end{aligned} \tag{16.10}$$

If $f(x) \in L^p(-\pi, \pi)$, then by (16.10)

$$\begin{aligned}
&\sum_{-N}^{N} e^{i\lambda_n x}\int_{-\pi}^{\pi} f(y)h_n(y)\,dy - \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin(N+\frac{1}{2})(x-y)}{x-y}\,dy \\
&\qquad = \frac{1}{4\pi^2 i}\int_{-\pi}^{\pi} f(y)\,dy \operatorname*{l.i.m.}_{A\to\infty}\int_{-A}^{A} G(u)e^{-iuy}\,du\int_C \frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)}\,d\zeta.
\end{aligned} \tag{16.11}$$

We want to show that the right side of equation (16.11) tends to zero as $N \to \infty$. Let

$$\begin{aligned}
I_1(x) = \Bigg| &\int_{-\pi}^{\pi} f(y)\,dy \operatorname*{l.i.m.}_{A\to\infty}\int_{-A}^{A} G(u)e^{-iuy}\,du \\
&\cdot\int_{-N-1/2}^{N+1/2}\frac{e^{i\xi x - Mx}}{G(\xi+iM)(u-\xi-iM)}\,d\xi\Bigg|.
\end{aligned} \tag{16.12}$$

This is the absolute value of that part of the right side of (16.11) for which ζ varies over the upper horizontal side of the rectangle C. By (16.03)

$$G(u) = O(|u|^{4D}).$$

Since $4D < 1$ it follows that

$$\frac{G(u)}{(u+i)^2} \in L(-\infty, \infty). \tag{16.13}$$

Using

$$\frac{1}{u-\xi-iM} = \frac{1}{u-iM} + \frac{\xi}{(u-iM)(u-\xi-iM)} \tag{16.14}$$

in (16.12), we obtain

$$I_1(x) = \Bigl| \int_{-\pi}^{\pi} f(y)\,dy \ \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} \frac{G(u)e^{-iuy}}{u - iM}\,du \int_{-N-1/2}^{N+1/2} \frac{e^{i\xi x - Mx}}{G(\xi + iM)}\,d\xi \tag{16.15}$$
$$+ \int_{-\pi}^{\pi} f(y)\,dy \int_{-\infty}^{\infty} \frac{G(u)e^{-iuy}}{u - iM}\,du \int_{-N-1/2}^{N+1/2} \frac{\xi e^{i\xi x - Mx}}{G(\xi + iM)(u - \xi - iM)}\,d\xi \Bigr|.$$

By (16.13) the second term on the right above is absolutely integrable.

Let $g(u)/(2\pi)^{1/2}$ be the Fourier transform of $f(y)$. Since $f(y) \in L^p$ and since

$$\int_{-A}^{A} \frac{G(u)e^{-iuy}}{u - iM}\,du$$

converges in the qth mean by Theorem G, where $q = p/(p-1)$, we can invert the order of integration in the following integral giving

$$\int_{-\pi}^{\pi} f(y)\,dy \ \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{A} \frac{G(u)e^{-iuy}}{u - iM}\,du$$
$$= \int_{-\infty}^{\infty} \frac{G(u)}{u - iM}\,du \int_{-\pi}^{\pi} f(y)e^{-iuy}\,dy = \int_{-\infty}^{\infty} \frac{G(u)g(u)}{u - iM}\,du.$$

(16.15) now becomes

$$I_1(x) = \Bigl| \int_{-\infty}^{\infty} \frac{G(u)g(u)}{u - iM}\,du \int_{-N-1/2}^{N+1/2} \frac{e^{i\xi x - Mx}}{G(\xi + iM)}\,d\xi$$
$$+ \int_{-\infty}^{\infty} \frac{G(u)g(u)}{u - iM}\,du \int_{-N-1/2}^{N+1/2} \frac{\xi e^{-Mx + i\xi x}}{G(\xi + iM)(u - \xi - iM)}\,d\xi \Bigr|$$

where we have also inverted the order of integration in the second term on the right of (16.15). Or, using (16.14),

$$I_1(x) = \Bigl| \int_{-\infty}^{\infty} G(u)g(u)\,du \int_{-N-1/2}^{N+1/2} \frac{e^{-Mx + i\xi x}}{G(\xi + iM)(u - \xi - iM)}\,d\xi \Bigr|. \tag{16.16}$$

(All we have done in going from (16.12) to (16.16) is justify an inversion of order of integration.)

Clearly for $M > 1$ and $|\xi| < 2N$,

$$\Bigl| \frac{G(u)}{u - \xi - iM} \Bigr| < A_{14} N \Bigl| \frac{G(u)}{u - \lambda_0} \Bigr|.$$

Thus (16.16) becomes

$$I_1(x) \leqq A_{14} N e^{M|x|} \int_{-N-1/2}^{N+1/2} \frac{d\xi}{|G(\xi + iM)|} \int_{-\infty}^{\infty} \Bigl| \frac{G(u)g(u)}{u - \lambda_0} \Bigr|\,du. \tag{16.17}$$

Using Holder's inequality,

$$\int_{-\infty}^{\infty} \Bigl| \frac{G(u)g(u)}{u - \lambda_0} \Bigr|\,du \leqq \Bigl[\int_{-\infty}^{\infty} \Bigl| \frac{G(u)}{(u - \lambda_0)} \Bigr|^p du \Bigr]^{1/p} \Bigl[\int_{-\infty}^{\infty} |g(u)|^q\,du \Bigr]^{1/q}.$$

Using Theorem XX and Theorem G on Fourier transforms of functions in L^p,

$$\int_{-\infty}^{\infty}\left|\frac{G(u)g(u)}{u-\lambda_0}\right|du \leqq \frac{A_{15}}{(p-1-2pD)^2}\left[\int_{-\pi}^{\pi}|f(y)|^p\,dy\right]^{1/p}.$$

Using this result, (16.17) becomes

$$I_1(x) \leqq BNe^{M|x|}\int_{N-1/2}^{N+1/2}\frac{d\xi}{|G(\xi+iM)|} \tag{16.18}$$

where B is some constant depending only on $f(x)$, p, and D. By (16.04)

$$|G(\xi+iM)| \geqq A_{16}M(M^2+N^2)^{-1/2}{}^{-2D}e^{\pi M}, \qquad |\xi| \leqq N+\tfrac{1}{2}.$$

Using this in (16.18),

$$I_1(x) \leqq A_{18}BN^2(M^2+N^2)e^{-M(\pi-|x|)}. \tag{16.19}$$

Again let us consider that part of the right member of (16.11) for which ζ runs over the right vertical side of the rectangle C. We denote this by

$$I_2(x) = \left|\int_{-\pi}^{\pi}f(y)\,dy\ \underset{A\to\infty}{\mathrm{l.i.m.}}\int_{-A}^{A}G(u)e^{-iuy}\,du\right.$$
$$\left.\cdot\int_{-M}^{M}\frac{e^{-x\eta}}{G(N+\frac{1}{2}+i\eta)(u-N-\frac{1}{2}-i\eta)}\,d\eta\right|.$$

If we split the range of the integral in u into the ranges $(-A, 2N)$, $(-2N, 2N)$, and $(2N, A)$, then, proceeding in a manner similar to that used with $I_1(x)$, we can invert the order of integration in y and u, where in this case we use

$$\frac{1}{u-N-\frac{1}{2}-i\eta} = \frac{1}{u-\lambda_0} + \frac{N+\frac{1}{2}+i\eta-\lambda_0}{(u-\lambda_0)(u-N-\frac{1}{2}-i\eta)}$$

on the ranges $(-A, -2N)$, $(2N, A)$. If we define $g(u)$ as before, we have

$$I_2(x) \leqq \left|\int_{-\infty}^{\infty}G(u)g(u)\,du\int_{-M}^{M}\frac{e^{-x\eta}}{G(N+\frac{1}{2}+i\eta)(u-N-\frac{1}{2}-i\eta)}\,d\eta\right|.$$

Or making an obvious change of variable

$$I_2(x) \leqq \left|\int_{-\infty}^{\infty}g(u+N+\tfrac{1}{2})\,du\int_{-M}^{M}\frac{G(u+N+\frac{1}{2})e^{-x\eta}}{G(i\eta+N+\frac{1}{2})(u-i\eta)}\,d\eta\right|.$$

Let $\mu_n = -N + \lambda_{N+n}$. Then

$$|\mu_n - n| \leqq D$$

and

$$\left|\frac{G(u+N+\frac{1}{2})}{G(i\eta+N+\frac{1}{2})}\right| = \left(\frac{u+\frac{1}{2}-\mu_0}{i\eta+\frac{1}{2}-\mu_0}\right)\prod_1^{\infty}\left(\frac{u+\frac{1}{2}-\mu_n}{i\eta+\frac{1}{2}-\mu_n}\right)\left(\frac{u+\frac{1}{2}-\mu_{-n}}{i\eta+\frac{1}{2}-\mu_{-n}}\right).$$

Let

$$G_N(u) = (u - \mu_0) \prod_1^\infty \left(1 - \frac{u}{\mu_n}\right)\left(1 - \frac{u}{\mu_{-n}}\right).$$

Then using the two equations above,

$$I_2(x) \leqq \left| \int_{-\infty}^{\infty} g(u + N + \tfrac{1}{2}) G_N(u + \tfrac{1}{2})\, du \int_{-M}^{M} \frac{e^{-x\eta}}{G_N(i\eta + \tfrac{1}{2})(u - i\eta)}\, d\eta \right|.$$

Clearly

$$\left| \frac{1}{u - i\eta} \right| < \frac{4}{|u + \frac{1}{2} - \mu_0|}$$

unless $|u| < 1$ and $|\eta| < 1$, and therefore

$$I_2(x) \leqq 4 \int_{-\infty}^{\infty} \left| \frac{g(u + N + \frac{1}{2}) G_N(u + \frac{1}{2})}{u + \frac{1}{2} - \mu_0} \right| du \int_{-\infty}^{\infty} \frac{e^{|x\eta|}}{|G_N(\frac{1}{2} + i\eta)|}\, d\eta$$
$$+ \int_{-1}^{1} |g(u + N + \tfrac{1}{2}) G_N(u + \tfrac{1}{2})|\, du \left| \int_{-1}^{1} \frac{e^{-x\eta}}{(u - i\eta) G_N(\frac{1}{2} + i\eta)}\, d\eta \right|.$$

Since $G_N(w)$ satisfies exactly the same conditions as $G(w)$ the results that have been proved for $G(w)$ also hold for $G_N(w)$. Thus from inequalities (16.04) and (16.05)

$$\frac{e^{|x\eta|}}{|G_N(\frac{1}{2} + i\eta)|} < A_{19}(1 + |\eta|) e^{-(\pi - |x|)|\eta|}.$$

And therefore for $|x| < \pi$

$$(16.20)\qquad I_2(x) < \frac{A_{20}}{(\pi - |x|)^2} \int_{-\infty}^{\infty} \left| \frac{g(u + N + \frac{1}{2}) G_N(u + \frac{1}{2})}{u + \frac{1}{2} - \mu_0} \right| du$$
$$+ \int_{-1}^{1} |g(u + N + \tfrac{1}{2}) G_N(u + \tfrac{1}{2})|\, du \left| \int_{-1}^{1} \frac{e^{-x\eta}}{(u - i\eta) G_N(i\eta + \frac{1}{2})}\, d\eta \right|.$$

For $|\eta| \leqq 1$,

$$\left| \frac{e^{-x\eta}}{G_N(i\eta + \frac{1}{2})} - \frac{1}{G_N(\frac{1}{2})} \right| \leqq |\eta| \max_{|\eta| \leqq 1} \left| \frac{d}{d\eta} \frac{e^{-x\eta}}{G_N(i\eta + \frac{1}{2})} \right|.$$

Using (16.05) of Lemma 16.1 we have

$$\left| \frac{e^{-x\eta}}{G_N(i\eta + \frac{1}{2})} - \frac{1}{G_N(\frac{1}{2})} \right| \leqq |\eta| \left(\frac{\pi e^\pi}{A_8} + \frac{e^\pi}{A_8^2} \max_{|\eta| \leqq 1} |G_N'(i\eta + \tfrac{1}{2})| \right).$$

But

$$G_N'(i\eta + \tfrac{1}{2}) = \frac{1}{2\pi i} \int_T \frac{G_N(z)}{(z - i\eta + \frac{1}{2})^2}\, dz$$

where T can be taken as a circle of radius 10 about the origin. Using (16.03) of Lemma 16.1, we see that there exists an A_{21} such that $|G_N'(i\eta + \frac{1}{2})| < A_{21}$,

$|\eta| < 1$. Therefore

$$\left| \frac{e^{-x\eta}}{G_N(\frac{1}{2} + i\eta)} - \frac{1}{G_N(\frac{1}{2})} \right| \leqq A_{22}|\eta|, \qquad |\eta| \leqq 1.$$

Using this we have

$$\left| \int_{-1}^{1} \frac{e^{-x\eta}}{(u - i\eta)G_N(i\eta + \frac{1}{2})} d\eta \right| \leqq \frac{1}{|G_N(\frac{1}{2})|} \left| \int_{-1}^{1} \frac{d\eta}{\eta + iu} \right| + A_{22} \int_{-1}^{1} \frac{|\eta|}{|\eta + iu|} d\eta$$

$$\leqq \frac{\pi}{A_8} + 2A_{22} < A_{23}.$$

Using this in (16.20) we have

$$I_2(x) \leqq \frac{A_{20}}{(\pi - |x|)^2} \int_{-\infty}^{\infty} \left| \frac{g(u + N + \frac{1}{2})G_N(u + \frac{1}{2})}{u + \frac{1}{2} - u_0} \right| du$$

$$+ A_{23} \int_{-1}^{1} |g(u + N + \tfrac{1}{2})| \, du.$$

If we now split the range of integration $(-\infty < u < \infty)$ above into two parts and apply Holder's inequality, we get

$$I_2(x) \leqq \frac{A_{20}}{(\pi - |x|)^2} \left[\int_{-\infty}^{-N/2} \left| \frac{G_N(u + \frac{1}{2})}{u + \frac{1}{2} - \mu_0} \right|^p du \right]^{1/p} \left[\int_{-\infty}^{-N/2} |g(u + N + \tfrac{1}{2})|^q du \right]^{1/q}$$

$$+ \frac{A_{20}}{(\pi - |x|)^2} \left[\int_{-N/2}^{\infty} \left| \frac{G_N(u + \frac{1}{2})}{u + \frac{1}{2} - \mu_0} \right|^p du \right]^{1/p} \left[\int_{-N/2}^{\infty} |g(u + N + \tfrac{1}{2})|^q du \right]^{1/q}$$

$$+ 2A_{23} \left[\int_{-1}^{1} |g(u + N + \tfrac{1}{2})|^q du \right]^{1/q}.$$

Or

$$I_2(x) \leqq \frac{A_{20}}{(\pi - |x|)^2} \left[\int_{-\infty}^{N/2} \left| \frac{G_N(u + \frac{1}{2})}{u + \frac{1}{2} - \mu_0} \right|^p du \right]^{1/p} \left[\int_{-\infty}^{\infty} |g(u)|^q du \right]^{1/q}$$

$$\text{(16.21)} \qquad + \frac{A_{20}}{(\pi - |x|)^2} \left[\int_{N/2}^{\infty} |g(u + \tfrac{1}{2})|^q du \right]^{1/q} \left[\int_{-\infty}^{\infty} \left| \frac{G_N(u + \frac{1}{2})}{u + \frac{1}{2} - \mu_0} \right|^p du \right]^{1/p}$$

$$+ 2A_{23} \left[\int_{N/2}^{\infty} |g(u + \tfrac{1}{2})|^q du \right]^{1/q}.$$

By Theorem XX

$$\int_{-\infty}^{N/2} \left| \frac{G_N(u + \frac{1}{2})}{u + \frac{1}{2} - \mu_0} \right| du \leqq \frac{A_{24} N^{-(p-1-2pD)}}{(p - 1 - 2pD)^2}.$$

Also we recall that $g(u) \in L^q$ and therefore

$$\lim_{N \to \infty} \int_{N/2}^{\infty} |g(u + \tfrac{1}{2})|^q du = 0.$$

Using these results in (16.21), we get

(16.22) $$\lim_{N\to\infty} I_2(x) = 0, \qquad |x| \leqq \pi - \delta,\ \delta > 0,$$

uniformly in x. This holds entirely independently of M.

Let us denote that part of the integral on the right side of (16.11) for which ζ varies along the lower horizontal side of the rectangle C by $I_3(x)$. Then clearly $I_3(x)$ satisfies the same inequality as $I_1(x)$, (16.19). We also introduce $I_4(x)$, similarly related to $I_2(x)$.

By (16.22), we can make $I_2(x)$, and therefore $I_2(x) + I_4(x)$, arbitrarily small independently of M by choosing N sufficiently large. Having done this we can for the particular value of N, make $I_1(x) + I_3(x)$ arbitrarily small by making M sufficiently large, as is clear from (16.19). Thus (16.11) becomes

$$\lim_{N\to\infty}\left[\sum_{-N}^{N} e^{i\lambda_n x}\int_{-\pi}^{\pi} f(y)h_n(y)\,dy - \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin (N+\frac{1}{2})(x-y)}{x-y}\,dy\right] = 0$$

uniformly in x for $|x| \leqq \pi - \delta$, $\delta > 0$. From this it follows by well-known results of Fourier series that

$$\lim_{N\to\infty}\sum_{-N}^{N}\left\{e^{i\lambda_n x}\int_{-\pi}^{\pi} f(y)h_n(y)\,dy - \frac{e^{inx}}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\,dy\right\} = 0.$$

That the sequence $\{e^{i\lambda_n x}\}$ is closed $L^p(-\pi, \pi)$ follows at once from Theorem IV, (3.03). Since $\{e^{i\lambda_n x}\}$ is closed, it follows that $\{h_n(x)\}$ is unique. Thus to complete the proof of Theorem XVIII we have only to show that the summability properties of non-harmonic Fourier series are the same as for ordinary Fourier series. To show this requires a slight variation of the ideas used above. We shall take the case of Riesz summability of order α How to proceed with various other types of summability will be obvious from this. We suppose that $\lambda_0 \neq 0$.

Let C_1 represent the rectangle with vertices at $(-iM,\ N + \frac{1}{2} - iM,\ N + \frac{1}{2} + iM,\ iM)$ and let C_2 be the reflection of C_1 in the imaginary axis. Then as in (16.10)

(16.23)
$$\begin{aligned}\frac{1}{4\pi^2 i}\,\underset{A\to\infty}{\text{l.i.m.}}\int_{-A}^{A} G(u)e^{-iuy}\,du\Bigg[\int_{C_1}\frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)}\left(1-\frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha}d\zeta \\ + \int_{C_2}\frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)}\left(1+\frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha}d\zeta\Bigg] \\ = \sum_{-N}^{N} h_n(y)e^{i\lambda_n x}\left(1-\frac{|\lambda_n|}{N+\frac{1}{2}}\right)^{\alpha} - \frac{1}{\pi}\int_0^1 \cos\frac{u(x-y)}{N+\frac{1}{2}}(1-u)^{\alpha}\,du.\end{aligned}$$

If we now multiply through by $f(y)$ and integrate as in (16.11), we have to show that the repeated integral analogous to the right side of (16.11) tends uniformly to zero as $N \to \infty$ if $|x| \leqq \pi - \delta$. That this is true is clear if we observe that the integral from $\zeta = iM$ to $\zeta = -iM$ in C_1 is cancelled by the integral from $\zeta = -iM$ to $\zeta = iM$ in C_2 in (16.23) and thus we are really only

concerned with ζ as it varies over the sides of the rectangle C in (16.13). This we can handle in a manner similar to that used on (16.11). Thus

$$\lim_{N\to\infty}\sum_{-N}^{N}\left[e^{i\lambda_n x}\left(1-\frac{|\lambda_n|}{N+\frac{1}{2}}\right)^{\alpha}\int_{-\pi}^{\pi} f(y)h_n(y)\,dy\right.$$

$$\left.-e^{inx}\left(1-\frac{|n|}{N+\frac{1}{2}}\right)^{\alpha}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\,dy\right]=0$$

uniformly for $|x| \leqq \pi - \delta$ if $\lambda_0 \neq 0$.

If $\lambda_0 = 0$ we can modify the contours C_1 and C_2 by taking a small semi-circular path of radius ρ around the origin. In this case we have to consider not only that part of (16.23) for which ζ varies over C but also a part involving

$$\int_{-\pi/2}^{\pi/2}\frac{e^{i\zeta x}}{G(\zeta)(u-\zeta)}\left[\left(1-\frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha}-\left(1+\frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha}\right]\rho e^{i\theta}\,d\theta$$

where $\zeta = \rho e^{i\theta}$. The fact that

$$\left(1-\frac{\zeta}{N+\frac{1}{2}}\right)^{\alpha}-\left(1+\frac{\zeta}{N+1}\right)^{\alpha}$$

has a zero at $\zeta = 0$ is what allows us to discard this part of the left side of (16.23).

Thus the Riesz summability of the two series are the same. The same method applies to Abel and similar types of summability.

Proof of Theorem XIX. Here we show that the set

$$
\begin{aligned}
&\lambda_n = n - \frac{p-1}{2p}, && n > 0,\\
(16.24)\qquad &\lambda_0 = 0,\\
&\lambda_{-n} = -n + \frac{p-1}{2p}, && n > 0,
\end{aligned}
$$

which obviously satisfies (15.05) does not have the properties which we have demonstrated in Theorem XVIII for sets satisfying (15.03).

Let $q = p/(p-1)$, and let us consider $\cos^{1/q}\frac{1}{2}x$. Clearly for $n \geqq 1$

$$
\begin{aligned}
\int_{-\pi}^{\pi} e^{i\lambda_n x}\cos^{-1/q}\tfrac{1}{2}x\,dx &= \int_{-\pi}^{\pi} e^{ix(n-1/2q)}\left(\frac{e^{ix/2}+e^{-ix/2}}{2}\right)^{-1/q}dx\\
&= 2^{1/q}\int_{-\pi}^{\pi} e^{inx}(1+e^{ix})^{-1/q}\,dx\\
(16.25)\qquad &= 2^{1/q}\lim_{r\to 1-0}\int_{-\pi}^{\pi} e^{inx}(1+re^{ix})^{-1/q}\,dx\\
&= 2^{1/q}\lim_{r\to 1-0}\sum_{k=0}^{\infty} r^k\binom{-1/q}{k}\int_{-\pi}^{\pi} e^{i(n+k)x}\,dx.
\end{aligned}
$$

But for $n \geqq 1$, $k \geqq 0$ the integrals on the right above are all zero. Thus (16.25)

vanishes for $n \geqq 1$. Since $\lambda_{-n} = -\lambda_n$, by taking the conjugate of the first expansion in (16.25) it is clear that it also vanishes for $n \leqq -1$. Thus

$$\int_{-\pi}^{\pi} e^{i\lambda_n x} \cos^{-1/q} \tfrac{1}{2}x \, dx = 0, \qquad n \neq 0. \tag{16.26}$$

Also since $\cos \frac{1}{2}x > 0$, $|x| < \pi$, there must exist an a such that

$$a \int_{-\pi}^{\pi} \cos^{-1/q} \tfrac{1}{2}x \, dx = 1. \tag{16.27}$$

If $p < 2$ then $q > 2$ and

$$\int_{-\pi}^{\pi} | \cos^{-1/q} \tfrac{1}{2}x |^p \, dx < \infty. \tag{16.28}$$

If (15.05) is sufficient for Theorem XVIII, it follows that there must exist an $h_0(x)$ such that

$$\int_{-\pi}^{\pi} e^{i\lambda_n x} h_0(x) \, dx = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases} \tag{16.29}$$

By Theorem IV, (3.03), it is clear that $\{e^{i\lambda_n x}\}$ given by (16.24) is closed $L^p(-\pi, \pi)$. But by (16.26), (16.27), and (16.29),

$$\int_{-\pi}^{\lambda} e^{i\lambda_n x} \{h_0(x) - a \cos^{1/q} \tfrac{1}{2}x\} \, dx = 0, \qquad -\infty < n < \infty.$$

Thus $h_0(x) - a \cos^{-1/q} \frac{1}{2}x$ vanishes almost everywhere, and therefore

$$h_0(x) = a \cos^{-1/q} \tfrac{1}{2}x. \tag{16.30}$$

Clearly if

$$f(x) = \begin{cases} 0, & |x| < \frac{1}{2}\pi, \\ \dfrac{1}{(\pi - |x|)^{1/p} |\log (\pi - |x|)|}, & \frac{1}{2}\pi < |x| < \pi, \end{cases} \tag{16.31}$$

then $f(x) \in L^p(-\pi, \pi)$. But

$$\int_{-\pi}^{\pi} h_0(x) f(x) \, dx > a \int_{\pi/2}^{\pi} \frac{\cos^{-1/q} \frac{1}{2}x}{(\pi - x)^{1/p} |\log (\pi - x)|} \, dx$$

$$> a \int_{\pi/2}^{\pi} \frac{2}{(\pi - x)^{1/q} (\pi - x)^{1/p} |\log (\pi - x)|} \, dx = \infty,$$

since

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Thus if $p < 2$ there exist functions $f(x)$ which cannot be expanded into a non-harmonic Fourier series in the set (16.24). That is, Theorem XVIII is not valid for this set.

In case $p = 2$ then (16.28) does not hold and $h_0(x)$ must be identified with $a \cos^{1/q} \frac{1}{2}x$ by a different method. Assuming that (16.24) is sufficient for Theorem XVIII in case $p = 2$ there exists an $h_0(x)$ such that if

$$\int_{-\pi}^{\pi} h_0(x)e^{iwx}\,dx = H_0(w)$$

then

$$H_0(0) = 1, \qquad H_0(\lambda_n) = 0, \qquad\qquad n \neq 0.$$

Since $\lambda_{-n} = -\lambda_n$ in (16.24), the function $\frac{1}{2}(h_0(x) + h_0(-x))$ has the same orthogonality properties as $h_0(x)$. Since $h_0(x)$ is unique, it must therefore be even. Thus $H_0(w)$ is even. Let

$$F(w) = \prod_1^\infty \left(1 - \frac{w^2}{\lambda_n^2}\right).$$

Since $H_0(w)$ vanishes at all the zeros of $F(w)$,

$$H_0(w) = \phi(w)F(w).$$

But exactly as in the proof of Theorem III it can be shown that $\phi(w)$ has no zeros. Thus

$$H_0(w) = e^{cw}F(w).$$

But $H_0(w)$ and $F(w)$ are even. Therefore $c = 0$ and

$$H_0(w) = F(w).$$

If

$$H_1(w) = a \int_{-\pi}^{\pi} \cos^{1/2} \tfrac{1}{2}x e^{iwx}\,dx,$$

then by (16.26), $H_1(w)$ vanishes at the zeros of $F(w)$, and, exactly as with $H_0(w)$, it follows that

$$H_1(w) = F(w).$$

Thus

$$H_0(w) = H_1(w).$$

But this is possible only if

$$h_0(x) = a \cos^{-1/2} \tfrac{1}{2}x.$$

That is, (16.30) holds for $p = 2$. The contradiction now follows in the same way as for $p < 2$.

17. Proof of a related inequality of Hardy and Levinson. A more precise inequality than that of Lemma 15.2 has been given.[3]

[3] Hardy and Levinson, *Inequalities satisfied by a certain definite integral*, Bulletin of the American Mathematical Society, vol. 43 (1937), p. 709.

THEOREM XXI. *Let*

$$0 \leqq a_1 < a_2 < \quad < a_{2n+1} \leqq 1,$$

(17.01) $$f(x) = \frac{(x - a_2)(x - a_4) \cdots (x - a_{2n})}{(x - a_1)(x - a_3) \cdots (x - a_{2n+1})},$$

$$J(t) = \int_0^1 |f(x)|^t \, dx$$

Then

$$\frac{\Gamma(\frac{1}{2} + \frac{1}{2}t)\Gamma(1 - \frac{1}{2}t)}{(1 - t)\pi^{1/2}} \leqq J(t) \leqq \frac{2^t}{1 - t}$$

with inequality except when

$$f(x) = \frac{1}{x - \frac{1}{2}}, \qquad J(t) = \frac{2^t}{1 - t},$$

$$f(x) = \frac{x - \frac{1}{2}}{x(x - 1)}, \qquad J(t) = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}t)\Gamma(1 - \frac{1}{2}t)}{(1 - t)\pi^{1/2}}$$

LEMMA 17.1. *If $f(x)$ satisfies (17.01), then*

(17.02) $$\int_{-\infty}^{\infty} F\{f(x)\} \, dx = \int_{-\infty}^{\infty} F(y) \frac{dy}{y^2}$$

whenever $F(y)$ is defined for all values of y and either integral exists as a Lebesgue integral.

Lemma 17.1 is essentially the same as a theorem of Boole.[4]

There are two other definitions of $f(x)$ equivalent to that of the relations (17.01). In the first place, as we can verify at once by resolving $f(x)$ into partial fractions,

(17.03) $$f(x) = \sum_{\nu=0}^{n} \frac{\alpha_\nu}{x - a_{2\nu+1}},$$

where

(17.04) $$\alpha_\nu > 0, \qquad \sum \alpha_\nu = 1.$$

This is the form which we shall generally use here. Secondly

$$g(x) = \frac{1}{f(x)} = x - \sum_{\nu=0}^{n} \frac{\beta_\nu}{x - a_{2\nu}}$$

where $\beta_\nu > 0$. If we write $1/y$ for y and $G(y)$ for $F(1/y)$, then (17.02) becomes

[4] G. Boole, *On the comparison of transcendents, with certain applications to the theory of definite integrals*, Philosophical Transactions of the Royal Society, vol. 147 (1857), pp. 745–803, in particular page 780.

$$\int_{-\infty}^{\infty} G\{g(x)\}\,dx = \int_{-\infty}^{\infty} G(y)\,dy,$$

which is Boole's formula.

To prove Lemma 17.1 we observe that, after (17.03) and (17.04), the graph of $f(x)$ consists of $n + 2$ descending pieces corresponding to the intervals $(-\infty, a_1)$, (a_1, a_3), $\cdots$, (a_{2n+1}, ∞), the corresponding intervals of variation of $f(x)$ being $(0, -\infty)$, $(\infty, -\infty)$, $\cdots$, $(\infty, 0)$; and that, when x moves from $-\infty$ to ∞, $f(x)$ moves, in all, $n + 1$ times over the same range. The line $f(x) = y$ cuts the graph of $f(x)$ in $n + 1$ points $x_1, x_2, \cdots, x_{n+1}$; and

$$\int_{-\infty}^{\infty} F(y)\,dx = \int_{-\infty}^{\infty} F(y)P(y)\frac{dy}{y^2},$$

where

$$P(y) = -y^2 \sum_\nu \left(\frac{dx}{dy}\right)_{x=x_\nu}.$$

We have to prove that

$$P(y) = 1.$$

It follows from considering $y - f(x)$ that

$$(17.05)\qquad y\prod_\nu (x - a_{2\nu+1}) - \sum_\nu \alpha_\nu \prod_{\mu\neq\nu} (x - a_{2\mu+1}) = y \prod_\nu (x - x_\nu),$$

where y on the right side comes from comparing the coefficients of x^n on each side of the equation. Hence, first, equating the coefficients of x^{n-1} in (17.05) and using (17.04), we have

$$(17.06)\qquad \sum x_\nu - \sum a_{2\nu+1} = \frac{1}{y}.$$

Next (17.06) is an identity in y when $x_\nu(y)$ is substituted for x_ν. Hence, differentiating this we obtain

$$\sum \frac{dx_\nu}{dy} = -\frac{1}{y^2}.$$

It follows that $P(y) = 1$. This completes the proof of the lemma.

In what follows it is convenient to symmetrize our analysis about the origin, which we can do by writing $x - \frac{1}{2}$ for x. We have then

$$(17.07)\qquad J(t) = \int_{-1/2}^{1/2} |f(x)|^t\,dx, \quad f(x) = \sum \frac{\alpha_\nu}{x - a_{2\nu-1}}, \quad \alpha_\nu > 0, \sum \alpha_\nu = 1,$$

$$(17.08)\qquad -\tfrac{1}{2} \leq a_1 < a_2 < \cdots < a_{2n+1} \leq \tfrac{1}{2}.$$

Lemma 17.2. *If $f(x)$ satisfies* (17.07) *and* (17.08), *then*

$$(17.09)\qquad J(t) = \frac{2^t}{1 - t} - \int_{1/2}^{\infty} \left\{ |f(x)|^t + |f(-x)|^t - \frac{2}{x^t} \right\} dx.$$

Proof of Lemma 17.2. Suppose that ϵ is small and positive and that ξ and η are the largest and smallest roots of $f(x) = \epsilon$ and $f(x) = -\epsilon$ respectively. Then $\xi > \frac{1}{2}$ and $\eta < -\frac{1}{2}$. Also

$$\frac{1}{\xi + \frac{1}{2}} \leqq \sum \frac{\alpha_\nu}{\xi - a_{2\nu+1}} = \epsilon \leqq \frac{1}{\xi - \frac{1}{2}},$$

and so

$$\frac{1}{\epsilon} - \frac{1}{2} \leqq \xi \leqq \frac{1}{\epsilon} + \frac{1}{2},$$

$$\xi = \frac{1}{\epsilon} + O(1), \tag{17.10}$$

where the O refers to the limit process $\epsilon \to 0$. Similarly

$$\eta = -\frac{1}{\epsilon} + O(1). \tag{17.11}$$

Define f_ϵ by the relations

$$f_\epsilon = f, \ (|f| \geqq \epsilon); \qquad f_\epsilon = 0, \ (|f| < \epsilon).$$

Then, by Lemma 17.1

$$\int_{-\infty}^{\infty} |f_\epsilon|^t \, dx = 2 \int_\epsilon^\infty |y|^{t-2} \, dy = \frac{2\epsilon^{t-1}}{1-t}.$$

Hence

$$\begin{aligned} J(t) &= \int_{-1/2}^{1/2} |f|^t \, dx = \lim_{\epsilon \to 0} \int_{-1/2}^{1/2} |f_\epsilon|^t \, dx \\ &= \lim_{\epsilon \to 0} \left\{ \frac{2\epsilon^{t-1}}{1-t} - \left(\int_{1/2}^{\xi} |f|^t \, dx + \int_\eta^{-1/2} |f|^t \, dx \right) \right\}. \end{aligned} \tag{17.12}$$

Now

$$f(x) = x^{-1} + O(x^{-2}), \qquad |f(x)|^t = |x|^{-t} + O(|x|^{-t-1})$$

for large x. Hence by (17.10)

$$\int_{1/\epsilon}^{\xi} |f|^t \, dx = \frac{1}{1-t} \left\{ \left(\frac{1}{\epsilon} + O(1) \right)^{1-t} - \left(\frac{1}{\epsilon} \right)^{1-t} \right\} + O(\epsilon^t) = O(\epsilon^t),$$

and we may replace ξ by $1/\epsilon$ in (17.12). Similarly we may replace η by $-1/\epsilon$. Hence

$$\begin{aligned} J(t) &= \lim_{\epsilon \to 0} \left\{ \frac{2\epsilon^{t-1}}{1-t} - \int_{1/2}^{1/\epsilon} \{ |f(x)|^t + |f(-x)|^t \} \, dx \right\} \\ &= \lim_{\epsilon \to 0} \left\{ \frac{2\epsilon^{t-1}}{1-t} - 2 \int_{1/2}^{1/\epsilon} \frac{dx}{x^t} - \int_{1/2}^{1/\epsilon} \left\{ |f(x)|^t + |f(-x)|^t - \frac{2}{x^t} \right\} dx \right\}, \end{aligned}$$

which is (17.09).

Lemma 17.3. *If* $|x| > \frac{1}{2}$ *then*

$$\phi(x) = |f(x)|^t + |f(-x)|^t$$

is (for every x) least and greatest when $f(x)$ is $1/x$ and $x/(x^2 - \frac{1}{4})$ respectively.

Proof of Lemma 17.3. We may suppose $x > \frac{1}{2}$. We consider the pole A of $f(x)$ nearest to an end of $(-\frac{1}{2}, \frac{1}{2})$. If we suppose, for example, that $A > 0$, then $A = a_{2n+1}$. If

$$\xi = \frac{1}{x-a}, \qquad \xi' = \frac{1}{x+a}, \qquad \Xi = \frac{1}{x-A}, \qquad \Xi' = \frac{1}{x+A},$$

then all these numbers are positive and

$$\frac{\Xi}{\Xi'} > \frac{\xi}{\xi'} \geqq 1 \tag{17.13}$$

for any positive pole a other than A. If

$$\psi(A) = \phi(x) = |f(x)|^t + |f(-x)|^t = \left(\sum \frac{\alpha}{x-a}\right)^t + \left(\sum \frac{\alpha}{x+a}\right)^t,$$

then

$$\frac{1}{t}\frac{d\psi(A)}{dA} = |f(x)|^{t-1}\frac{\mathrm{A}}{(x-A)^2} - |f(-x)|^{t-1}\frac{\mathrm{A}}{(x+A)^2},$$

where A is the α corresponding to A. This will be positive if

$$\left(\frac{\Xi}{\Xi'}\right)^2 > \left(\frac{\sum \alpha\xi}{\sum \alpha\xi'}\right)^{1-t},$$

and this is true on account of (17.13).

Hence we decrease $\phi(x)$ by moving A to the left, to the next pole, or to the origin if there is no other positive pole. Similarly, if A were negative, we should decrease $\phi(x)$ by moving A to the right. It follows by repetition of the argument that $\phi(x)$ is least when all the a's coincide at the origin, and $f(x) = 1/x$.

Similarly $\phi(x)$ is greatest when all the a's are at one of the ends of $(-\frac{1}{2}, \frac{1}{2})$. In this case

$$f(x) = \frac{\alpha}{x - \frac{1}{2}} + \frac{1-\alpha}{x+\frac{1}{2}} = \frac{x-\beta}{x^2 - \frac{1}{4}},$$

where $\beta = \alpha - \frac{1}{2}$, $0 \leqq \alpha \leqq 1$, $|\beta| \leqq \frac{1}{2}$. Finally

$$|x-\beta|^t + |x+\beta|^t < 2|x|^t$$

if $|x| > \frac{1}{2}$, $\beta \neq 0$, so that the true maximum of $\phi(x)$ occurs when

$$f(x) = \frac{x}{x^2 - \frac{1}{4}}.$$

Proof of Theorem XXI. We take the interval as $(-\frac{1}{2}, \frac{1}{2})$ so that the two

critical functions are

$$f_1(x) = \frac{1}{x}, \qquad f_2(x) = \frac{x}{x^2 - \frac{1}{4}}.$$

It follows from (17.09) and Lemma 17.3 that

$$J(t) \leqq \frac{2^t}{1-t},$$

with inequality unless $f = f_1$. Also

$$\int_{-1/2}^{1/2} |f|^t \, dx - \int_{-1/2}^{1/2} |f_2|^t \, dx$$
$$= \int_{1/2}^{\infty} (|f_2(x)|^t + |f_2(-x)|^t - |f(x)|^t - |f(-x)|^t) \, dx,$$

by (17.09), and the last integral is positive, by Lemma 17.3, unless $f = f_2$. Finally

$$\int_{-1/2}^{1/2} |f_2|^t \, dx = \int_{-1/2}^{1/2} \left| \frac{x}{x^2 - \frac{1}{4}} \right|^t dx = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}t)\Gamma(1 - \frac{1}{2}t)}{(1-t)\pi^{1/2}},$$

by an elementary calculation.

CHAPTER V

FOURIER TRANSFORMS OF NONVANISHING FUNCTIONS[1]

18. Introduction. If a function $f(x) \epsilon L(-\pi, \pi)$ and if it has a Fourier series

$$f(x) \sim \sum_0^\infty a_n e^{inx},$$

that is, if

$$a_{-n} = 0, \qquad n = 1, 2, \cdots,$$

then, obviously, $f(x)$ is the boundary function of a function analytic in the unit circle, and, being such a boundary function, $f(x)$ cannot be equivalent to zero over any interval, no matter how small, unless $f(x)$ is equivalent to zero in the whole interval $(-\pi, \pi)$.

This result can be immediately generalized to the case where

$$f(x) \sim \sum_{-\infty}^\infty a_n e^{inx}$$

with

$$a_{-n} = O(e^{-\delta n}), \qquad n \to \infty,$$

for some $\delta > 0$. For in this case $f(x)$ is the boundary function of

$$g(w) = \sum_{-\infty}^\infty a_n w^n,$$

which is clearly analytic in the ring $e^{-\delta} < |w| < 1$; and therefore again $f(x)$ cannot be equivalent to zero over any interval without being entirely equivalent to zero.

Suppose that we have a function $f(x)$ whose Fourier coefficients do not satisfy any condition of the form $a_n = O(e^{-\delta n})$ or $a_{-n} = O(e^{-\delta n})$. Then $f(x)$ is no longer the boundary function of an analytic function. The question now arises whether or not we can find conditions under which such $f(x)$ have the property of vanishing completely if they vanish over any interval. That we can follows from results of Chapter II. For example, a much simpler result than Theorem XII and an immediate consequence of it is: If $f(x) \epsilon L(-\pi, \pi)$ and has the Fourier series

$$\sum_{-\infty}^\infty a_n e^{inx},$$

[1] This chapter is not on gap or density theorems but is related to these. Moreover the results of this chapter are necessary for later use.

where

$$a_{-n} = O(e^{-\theta(n)}), \qquad n \to \infty,$$

for an increasing function $\theta(u)$ such that

$$\int_1^\infty \frac{\theta(u)}{u^2}\, du = \infty,$$

then $f(x)$ cannot vanish over any interval unless it vanishes over $(-\pi, \pi)$.

There is an analogous result in Fourier transforms.

THEOREM XXII.[2] *Let $F(u) \in L(-\infty, \infty)$ and let*

$$F(u) = O(e^{-\theta(u)}), \qquad u \to \infty, \tag{18.01}$$

where $\theta(u)$ is a positive increasing function such that

$$\int_1^\infty \frac{\theta(u)}{u^2}\, du = \infty. \tag{18.02}$$

Then $f(x)$, the Fourier transform of $F(u)$, cannot vanish over any interval unless it is identically zero.

Condition (18.01) is a one-sided condition. The theorem would of course be true if (18.01) held for $u < 0$ instead of for $u > 0$.

The condition (18.01) certainly will be satisfied if

$$F(u) = O(e^{-\delta u}), \qquad u > 0, \tag{18.03}$$

for some $\delta > 0$. In this case we can obtain the result of Theorem XXII quite easily. Let $z = x + iy$ and let

$$f(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F(u) e^{-iuz}\, du, \qquad 0 \leq y < \delta.$$

Clearly by (18.03)

$$|f(z)| \leq \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} |F(u)| e^{uy}\, du + \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{uy} O(e^{-\delta u})\, du.$$

Thus $f(z)$ exists for $0 \leq y < \delta$. Similarly $f'(z)$ exists for $0 < y < \delta$ and therefore $f(z)$ is analytic in the strip $(0 < y < \delta,\ -\infty < x < \infty)$. Thus $f(x)$ is either analytic or at least takes the boundary value of an analytic function. Therefore, by well-known results in function theory it cannot vanish over any interval unless it vanishes identically.

In this sense Theorem XXII is an extension of the fact that an analytic function, not identically zero, with $(-\infty, \infty)$ as a boundary cannot vanish over any interval of $(-\infty, \infty)$. The relationship with the general class of

[2] Cf. N. Levinson, *On a class of non-vanishing functions*, Proceedings of the London Mathematical Society, vol. 41 (1936), p. 393.

functions which are analytic in part of the half-plane above the boundary $(-\infty, \infty)$ is given by the following theorem.

THEOREM XXIII.[3] *Let $F(u)$ satisfy the requirements of Theorem XXII. Its Fourier transform*

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F(u) e^{-iux} \, du. \tag{18.04}$$

Let $g(z)$ be analytic for $(a \leqq x \leqq b, 0 < y \leqq \gamma)$ and continuous in $(a \leqq x \leqq b, 0 \leqq y \leqq \gamma)$. Suppose $f(x) = g(x)$ for $a < \alpha \leqq x \leqq \beta < b$. Then $f(x) = g(x)$ for $a \leqq x \leqq b$.

Theorem XXII is a special case of Theorem XXIII with $g(z) \equiv 0$. Thus it will suffice to prove Theorem XXIII. Another special case of Theorem XXIII which is useful for subsequent application is the following result.

THEOREM XXIV. *Let $F(u)$ satisfy the requirements of Theorem XXII. Then if $f(x)$, the Fourier transform of $F(u)$, coincides with an analytic function over some interval it coincides with the analytic function over its entire interval of analyticity on the x axis.*

The condition (18.01) can be considerably weakened. The condition $F(u) \in L(-\infty, \infty)$ can also be considerably modified without otherwise affecting these theorems. This is done in Theorems XXVII and XXVIII.

19. Proof of the basic theorem. In proving Theorem XXIII we shall require several lemmas.

LEMMA 19.1. *If $\theta(u)$ is a positive increasing function of u and if*

$$\int_1^{\infty} \log \left(e^{-u} + e^{-\theta(u)}\right) \frac{du}{u^2} > -\infty, \tag{19.01}$$

then

$$\int_1^{\infty} \frac{\theta(u)}{u^2} \, du < \infty. \tag{19.02}$$

Proof of Lemma 19.1. Let the set of intervals over which $\theta(u) > u$ be denoted by E. Then by (19.01)

$$\int_E \log e^{-u} \frac{du}{u^2} + \int_{C(E)} \log e^{-\theta(u)} \frac{du}{u^2} > -\infty,$$

where $C(E)$ is the set of points complementary to E in $(1, \infty)$. From this it follows at once that

$$\int_E \frac{du}{u} < \infty, \tag{19.03}$$

[3] Cf. N. Levinson, *A theorem relating non-vanishing and analytic functions*, Journal of Mathematics and Physics, vol. 16 (1938), p. 185.

$$\int_{C(E)} \frac{\theta(u)}{u^2}\, du < \infty. \tag{19.04}$$

Let the intervals of E be denoted by $\{a_n, b_n\}$. From the definition of E as the intervals where $\theta(u) > u$ it follows that

$$\theta(u) \leqq b_n, \qquad a_n < u < b_n.$$

Thus

$$\int_E \frac{\theta(u)}{u^2}\, du \leqq \sum^{(n)} \int_{a_n}^{b_n} \frac{b_n}{u^2}\, du = \sum^{(n)} \left(\frac{b_n}{a_n} - 1\right). \tag{19.05}$$

From (19.03),

$$\sum^{(n)} \log \frac{b_n}{a_n} < \infty. \tag{19.06}$$

But (19.06) implies that

$$\sum^{(n)} \left(\frac{b_n}{a_n} - 1\right) < \infty.$$

Thus (19.05) gives

$$\int_E \frac{\theta(u)}{u^2}\, du < \infty.$$

This and (19.04) give (19.02).

LEMMA 19.2. *If $g(z)$ is analytic in $(0 \leqq x \leqq 1, 0 < y \leqq \gamma < \frac{1}{2}\pi)$ and continuous for $(0 \leqq x \leqq 1, 0 \leqq y \leqq \gamma)$, and if*

$$J_1(u) = \int_0^1 \left(\frac{e^x - 1}{e - e^x}\right)^{iu} (e^x - 1)^2(e - e^x)^2 g(x)\, dx, \tag{19.07}$$

then

$$| J_1(u) | \leqq 10^6 e^{-\gamma u/2} \max | g(z) |, \qquad u > 0, \tag{19.08}$$

where in $\max | g(z) |$, *z ranges over* $(0 \leqq x \leqq 1, 0 \leqq y \leqq \frac{1}{2}\gamma)$.

Proof of Lemma 19.2. Using the Cauchy integral theorem in (19.07) we take instead of the path $(z = x, 0 \leqq x \leqq 1)$ the path made up of the arms of an isosceles triangle $(z = x + i\gamma x, 0 \leqq x \leqq \frac{1}{2})$ and $(z = x + i\gamma(1 - x), \frac{1}{2} \leqq x \leqq 1)$. Thus

$$\begin{aligned} J_1(u) = (1 + i\gamma) \int_0^{1/2} (e^{x(1+i\gamma)} - 1)^{iu+2}(e - e^{x(1+i\gamma)})^{2-iu} g(x + i\gamma x)\, dx \\ + (1 - i\gamma) \int_{1/2}^1 (e^{x+i\gamma(1-x)} - 1)^{iu+2}(e - e^{x+i\gamma(1-x)})^{2-iu} \\ \cdot g(x + i\gamma(1 - x))\, dx. \end{aligned} \tag{19.09}$$

Clearly, for $0 \leqq x \leqq \frac{1}{2}$

$$\operatorname{am}\left(\frac{e^{x(1+i\gamma)}-1}{e-e^{x(1+i\gamma)}}\right) = \operatorname{am}\,(e^{i\gamma x} - e^{-x}) - \operatorname{am}\,(e^{1-x} - e^{i\gamma x})$$

$$(19.10) \qquad = \tan^{-1}\frac{\sin \gamma x}{\cos \gamma x - e^{-x}} + \tan^{-1}\frac{\sin \gamma x}{e^{1-x} - \cos \gamma x}$$

$$\geqq \tan^{-1}\frac{\sin \gamma x}{\cos \gamma x - e^{-x}}.$$

But for $0 \leqq x \leqq \frac{1}{2}$

$$x \geqq \cos \gamma x - e^{-x},$$

since this inequality holds for $x = 0$ and

$$\frac{d}{dx}(x - \cos \gamma x + e^{-x}) = 1 + \gamma \sin \gamma x - e^{-x} \geqq 0$$

for $0 < \gamma < \frac{1}{2}\pi$, $0 \leqq x \leqq \frac{1}{2}$. Thus (19.10) becomes

$$\operatorname{am}\left(\frac{e^{x(1+i\gamma)}-1}{e-e^{x(1+i\gamma)}}\right) \geqq \tan^{-1}\frac{\sin \gamma x}{x}$$

$$\geqq \tan^{-1}\frac{\sin \frac{1}{2}\gamma}{\frac{1}{2}}, \qquad 0 \leqq x \leqq \tfrac{1}{2},$$

$$= \tan^{-1}(2 \cos \tfrac{1}{2}\gamma \tan \tfrac{1}{2}\gamma)$$

$$\geqq \tan^{-1}\left(\frac{2}{2^{1/2}} \tan \tfrac{1}{2}\gamma\right) > \tan^{-1}(\tan \tfrac{1}{2}\gamma) = \tfrac{1}{2}\gamma.$$

Thus for $u > 0$,

$$\left|(1 + i\gamma)\int_0^{1/2}\left(\frac{e^{x(1+i\gamma)}-1}{e-e^{x(1+i\gamma)}}\right)^{iu}(e^{x+i\gamma x} - 1)^2(e - e^{x(1+i\gamma)})^2 g(x + i\gamma x)\,dx\right|$$

$$\leqq \pi\tfrac{1}{2}e^{-\gamma u/2}(2e)^4 \max|g(z)| \leqq 2^{13}e^{-\gamma u/2}\max|g(z)|.$$

In precisely the same manner this holds for the second integral on the right of (19.09). Thus

$$|J_1(u)| \leqq 2^{14}e^{-\gamma u/2}\max|g(z)|, \qquad u > 0.$$

This completes the proof of the lemma.

Lemma 19.3. *Let*

$$(19.11) \qquad J_2(u, v) = \int_0^1 (e^x - 1)^{iu+2}(e - e^x)^{-iu+2}e^{-ivx}\,dx.$$

Then for $u > 0$ and any γ such that $0 < \gamma < \frac{1}{2}\pi$

$$|J_2(u, v)| \leqq 10^5 e^{-\gamma(u-v)/2}, \qquad v > 0, \tag{19.12}$$

$$|J_2(u, v)| < 10^5 e^{-\gamma u/2}, \qquad v < 0. \tag{19.13}$$

Also there exists a constant C such that

$$|J_2(u, v)| < C \frac{1 + u^2}{1 + v^2}. \tag{19.14}$$

Proof of Lemma 19.3. If in Lemma 19.2 we set $g(z) = e^{-ivz}$, then, for $0 \leqq y \leqq \frac{1}{2}\gamma$,

$$|e^{-iv(x+iy)}| \leqq e^{\gamma v/2}, \qquad v > 0,$$

$$|e^{-iv(x+iy)}| \leqq 1, \qquad v < 0.$$

Thus (19.12) and (19.13) follow from Lemma 19.2. By (19.11),

$$|J_2(u, v)| \leqq e^4. \tag{19.15}$$

For $|v| > 1$, if we integrate (19.11) by parts twice, each time integrating the term e^{-ivx} and differentiating the other terms, (19.14) follows at once. That (19.14) is also true if $|v| \leqq 1$ is clear from (19.15).

We can now prove Theorem XXIII.

Proof of Theorem XXIII. We shall prove that $f(x) = g(x)$ for $\beta \leqq x \leqq b$. That this is true for $a \leqq x \leqq \alpha$ follows in exactly the same way. This case can also be reduced to the first by considering $f(-x)$ and $g(-x)$. Thus it suffices to prove the result only for $\beta \leqq x \leqq b$.

With no restriction we can assume $\alpha = 0$ and $b = 1$. (Then $0 < \beta < 1$.) A simple linear transformation of variables reduces all cases to this one. Thus in what follows $\alpha = 0$ and $b = 1$. Our hypothesis now is that $f(x) = g(x)$, $(0 \leqq x \leqq \beta)$, where $0 < \beta < 1$, and we shall prove that $f(x) = g(x)$, $(\beta \leqq x \leqq 1)$.

Let

$$h(w) = \int_0^1 \{f(x) - g(x)\}(e^x - 1)^{2+iw}(e - e^x)^{2-iw}\, dx. \tag{19.16}$$

But $f(x) = g(x)$, $0 \leqq x \leqq \beta$. Thus

$$h(w) = \int_\beta^1 \{f(x) - g(x)\}(e^x - 1)^{2+iw}(e - e^x)^{2-iw}\, dx.$$

If $w = u + iv$, then, for $v > 0$, $h(w)$ is analytic and, for $v \geqq 0$, $h(w)$ is continuous. Also, for $v \geqq 0$

$$|h(w)| \leqq (e^\beta - 1)^{-v}(e - e^\beta)^v e^4 \int_\beta^1 |f(x) - g(x)|\, dx.$$

If B is defined by

$$\frac{e - e^\beta}{e^\beta - 1} = e^B,$$

then

$$|h(w)| \leqq Ae^{Bv}, \qquad v \geqq 0, \tag{19.17}$$

where A will be used to represent any constant depending only on $f(x)$ and $g(z)$.

From (19.16) and the definition of $J_1(u)$,

$$h(u) = -J_1(u) + \int_0^1 f(x)(e^x - 1)^{2+iu}(e - e^x)^{2-iu}\,dx.$$

Using the definition of $f(x)$,

$$h(u) = -J_1(u) + \int_0^1 (e^x - 1)^{2+iu}(e - e^x)^{2-iu}\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} F(y)e^{-iyx}\,dy.$$

From the definition of $J_2(u, v)$,

$$h(u) = -J_1(u) + \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} F(y)J_2(u, y)\,dy. \tag{19.18}$$

Clearly

$$\left|\int_{-\infty}^{\infty} F(y)J_2(u, y)\,dy\right| \leqq \left(\int_{-\infty}^{u/2} + \int_{u/2}^{\infty}\right)|F(y)J_2(u, y)|\,dy. \tag{19.19}$$

From Lemma 19.3

$$|J_2(u, y)| \leqq 10^5e^{-\gamma u/4}, \qquad y < \tfrac{1}{2}u,\ u > 0.$$

Using this and (19.14), (19.19) gives

$$\left|\int_{-\infty}^{\infty} F(y)J_2(u, y)\,dy\right| \leqq Ae^{-\gamma u/4} + Au^2\int_{u/2}^{\infty}\frac{|F(y)|}{y^2}\,dy.$$

Recalling that $|J_1(u)| \leqq Ae^{-\gamma u/2}$, $(u > 0)$, (19.18) and the above inequality give

$$|h(u)| \leqq Au^2\left(e^{-\gamma u/4} + \int_{u/2}^{\infty}\frac{|F(y)|}{y^2}\,dy\right), \qquad u > 1. \tag{19.20}$$

But $F(y) = O(e^{-\theta(y)})$, $(y \to \infty)$. Thus

$$|h(u)| \leqq Au^2(e^{-\gamma u/4} + e^{-\theta(u/2)}), \qquad u > 1. \tag{19.21}$$

Or

$$\int_1^{\infty} \log|h(u)|\frac{du}{u^2} \leqq A + \int_1^{\infty}\log(e^{-\gamma u/4} + e^{-\theta(u/2)})\frac{du}{u^2}.$$

By Lemma 19.1 since $\int_1^{\infty}\theta(\frac{1}{2}u)u^{-2}\,du = \infty$,

$$\int_1^{\infty}\log(e^{-\gamma u/4} + e^{-\theta(u/2)})\frac{du}{u^2} = -\infty.$$

Thus

$$\int_1^\infty \log |h(u)| \frac{du}{u^2} = -\infty. \tag{19.22}$$

We now use an argument already used previously. Let us assume that $h(w)$ is not identically zero. Then by using Carleman's theorem, Theorem B, on $h(w)$ in the upper half-plane it follows that

$$0 < \frac{1}{2\pi}\left(\int_{-R}^{-1} + \int_1^R\right) \log |h(u)| \left(\frac{1}{u^2} - \frac{1}{R^2}\right) du$$
$$+ \frac{1}{\pi R}\int_0^\pi \log |h(Re^{i\theta})| \sin \theta \, d\theta + A.$$

Or

$$-\int_1^R \log^- |h(u)| \left(\frac{1}{u^2} - \frac{1}{R^2}\right) du \leqq \left(\int_1^R + \int_{-R}^{-1}\right) \log^+ |h(u)| \frac{du}{u^2}$$
$$+ \frac{2}{R}\int_0^\pi \log^+ |h(Re^{i\theta})| \, d\theta + A.$$

But $|h(w)| \leqq Ae^{Bv}$, $(|v| \geqq 0)$. Using this gives

$$-\int_1^R \log^- |h(u)| \left(\frac{1}{u^2} - \frac{1}{R^2}\right) du < A$$

or

$$-\int_1^{R/2} \log^- |h(u)| \frac{du}{2u^2} < A.$$

Since A is independent of R, it follows that

$$-\int_1^\infty \log^- |h(u)| \frac{du}{u^2} < \infty. \tag{19.23}$$

But this contradicts (19.22). Thus $h(w) \equiv 0$, and therefore

$$\int_0^1 \{f(x) - g(x)\} \left(\frac{e^x - 1}{e - e^x}\right)^{iu} (e^x - 1)^2(e - e^x)^2 \, dx = 0. \tag{19.24}$$

If we now set

$$\frac{e^x - 1}{e - e^x} = e^t,$$

then the interval $(0 < x < 1)$ maps over to the interval $(-\infty < t < \infty)$, and (19.24) becomes the Fourier transform of an integrable function. But only a function equivalent to zero can have a Fourier transform that vanishes identically. This means that the function obtained by mapping $f(x) - g(x)$ on the interval $(-\infty < t < \infty)$ vanishes almost everywhere. Thus $f(x) - g(x)$ vanishes almost everywhere on $(0, 1)$. But this proves our theorem since both functions are continuous.

For the next section the following theorem is necessary.

THEOREM XXV. *Theorem* XXIII (*and therefore also* XXII) *remains true if*

$$\int_1^\infty \frac{du}{u^2} \log \int_u^\infty |F(y)| \frac{dy}{y^2} = -\infty \tag{19.25}$$

replaces conditions (18.01) *and* (18.02).

Theorem XXV includes Theorem XXIII, for if $F(y) = O(e^{\theta(y)})$ where $\theta(u)$ satisfies (18.02) then

$$\int_u^\infty |F(y)| \frac{dy}{y^2} = O\left(e^{\cdot\theta(u)} \int_u^\infty \frac{dy}{y^2}\right) = O(e^{-\theta(u)}). \tag{19.26}$$

By (18.02) this leads to (19.25).

Proof of Theorem XXV. The proof follows that of Theorem XXIII with only one difference, which we shall point out. Here we define

$$\theta(u) = -\log \int_u^\infty |F(y)| \frac{dy}{y^2}. \tag{19.27}$$

By (19.25)

$$\int_1^\infty \frac{\theta(u)}{u^2}\, du = \infty.$$

Also $\theta(u)$ is monotone increasing. To go from (19.20) to (19.21) in the proof of Theorem XXIII with $\theta(u)$ defined as in (19.27), we use (19.27) instead of $F(y) = O(e^{\theta(y)})$. Otherwise the proofs are identical.

20. Proofs of related theorems. The following theorem serves two purposes. It shows that the results of this chapter are best possible and it is necessary for subsequent results.

THEOREM XXVI.[4] *Let $\phi(u)$ be an even and a positive increasing function of $|u|$. If*

$$\int_1^\infty \frac{\phi(u)}{u^2}\, du < \infty, \tag{20.01}$$

then given any $x_0 > 0$ there exists an entire function $H(w)$, not equivalent to zero, such that

$$|H(u)| \leqq \frac{e^{-\phi(u)}}{1 + u^4} \tag{20.02}$$

and such that the Fourier transform of $H(w)$, $h(x)$, vanishes outside of $|x| \leqq x_0$ and $h(0) \neq 0$.

This theorem shows that condition (18.02) is best possible, since $H(u)$ which

[4] This result is essentially due to Paley and Wiener, loc. cit., chap. 1.

satisfies (20.01) and (20.02) has a transform that not only vanishes over an interval but over all except a finite part of the line.

Proof of Theorem XXVI. Let

$$\phi_1(u) = \phi(2u) + \log(100 + 100u^4). \tag{20.03}$$

Then $\phi_1(u)$ is monotone increasing for $u > 0$. Let

$$\sigma(u, v) = \int_{-\infty}^{\infty} \frac{(v+1)\phi_1(\xi)}{(u-\xi)^2 + (v+1)^2}\,d\xi. \tag{20.04}$$

Since $\phi_1(\xi) \geqq 0$,

$$\sigma(u, v) \geqq 0, \qquad v > -1.$$

Since

$$\frac{v+1}{(u-\xi)^2 + (v+1)^2} = \Re\,\frac{i}{w - \xi + i},$$

it follows that $\sigma(u, v)$ is a harmonic function for $v > -1$. Since $\phi_1(u)$ is increasing and positive,

$$\begin{aligned}\sigma(u, v) &\geqq \int_{|u|}^{\infty} \frac{\phi_1(\xi)(v+1)}{(|u| - \xi)^2 + (v+1)^2}\,d\xi \\ &\geqq \phi_1(u)\int_{|u|}^{\infty} \frac{v+1}{(\xi - |u|)^2 + (v+1)^2}\,d\xi = \frac{\pi}{2}\,\phi_1(u).\end{aligned}$$

Thus

$$\sigma(u, v) > \phi_1(u).$$

Let $\tau(u, v)$ be a conjugate function to $\sigma(u, v)$ in the half-plane $v > -1$. Since $\sigma(u, v) + i\tau(u, v)$ is analytic for $v > -1$,

$$H_1(w) = \frac{e^{-\sigma(u,v) - i\tau(u,v)}}{(w+i)^2}$$

is analytic in the half-plane $v > -1$. Since $\sigma(u, v)$ is positive,

$$|H_1(w)| \leqq \frac{1}{|w+i|^2} \leqq \frac{1}{1 + |w|^2}, \qquad v \geqq 0. \tag{20.05}$$

Also since $\sigma(u, v) > \phi_1(u)$

$$|H_1(w)| \leqq e^{-\phi_1(u)}, \qquad v \geqq 0.$$

Using (20.03)

$$|H_1(w)| \leqq \frac{e^{-\phi(2u)}}{100(1 + u^4)}. \tag{20.06}$$

Let

$$h_1(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H_1(u) e^{-iux}\, du.$$

For $x < 0$, the path of integration on the right can be closed in the upper half-plane since by (20.05)

$$| H_1(w) e^{-iwx} | \leqq \frac{e^{vx}}{1 + | w |^2}.$$

But the integral of an analytic function around a closed contour is zero. Thus

$$h_1(x) = 0, \qquad x < 0.$$

Since $H_1(u)$ is not identically zero, $h_1(x)$ cannot vanish identically. Since $| H_1(u) | < 1/(1 + u^4)$, $h_1(x)$ is continuous. Thus there must be a point $x_1 > 0$ such that $h_1(x_1) \neq 0$ but such that $h(x) = 0$, $x \leqq x_1 - x_0$ Let $h_2(x) = h_1(x + x_1)$. Then

$$h_2(0) \neq 0; \qquad h_2(x) = 0, \qquad (x \leqq -x_0).$$

If

(20.07) $$H_2(w) = H_1(w) e^{-iwx_1},$$

then

$$h_2(x) = h_1(x + x_1) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H_1(u) e^{-iu(x+x_1)}\, du$$
$$= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H_2(u) e^{-iux}\, du.$$

That is, $H_2(u)$ is the Fourier transform of $h_2(x)$.

Let

$$h(x) = h_2(x) h_2(-x).$$

Then

$$h(0) \neq 0; \qquad h(x) = 0, \qquad (| x | \geqq x_0).$$

Since $H_2(-u)$ must be the transform of $h_2(-x)$,

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_2(u) e^{-iux}\, du \int_{-\infty}^{\infty} H_2(-v) e^{-ivx}\, dv.$$

Let $v = t - u$. Then

$$h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_2(u)\, du \int_{-\infty}^{\infty} H_2(u - t) e^{-itx}\, dt$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}\, dt \int_{-\infty}^{\infty} H_2(u) H_2(u - t)\, du.$$

Thus the Fourier transform of $h(x)$ is

$$H(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H_2(\xi) H_2(\xi - u)\, d\xi.$$

From this, (20.06), and (20.07), it follows that there exists for $v \leqq 0$

$$H(w) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H_2(\xi) H_2(\xi - w)\, d\xi.$$

For $v \geqq 0$, transforming the variable of integration,

$$H(w) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H_2(\xi + w) H_2(\xi)\, d\xi.$$

Thus for $v \geqq 0$,

$$| H(w) | \leqq \int_{-\infty}^{-u/2} | H_2(\xi + w) | \max_{\xi \leq -u/2} | H_2(\xi) |\, d\xi$$
$$+ \int_{-u/2}^{\infty} | H_2(\xi) | \max_{-u/2 \leq \xi} | H_2(\xi + w) |\, d\xi.$$

By (20.06) and (20.07) this becomes

$$| H(w) | \leqq \frac{e^{vx_1 - \phi(u)}}{100(1 + u^4/16)} \int_{-\infty}^{\infty} \frac{d\xi}{100(1 + \xi^4)} + \frac{e^{vx_1 - \phi(u)}}{100(1 + u^4/16)} \int_{-\infty}^{\infty} \frac{d\xi}{100(1 + \xi^4)}$$
$$\leqq \frac{e^{|v|x_1 - \phi(u)}}{50 + 2u^4}, \qquad v \geqq 0.$$

Similarly this holds for $v \leqq 0$. Thus

$$(20.08) \qquad | H(w) | \leqq \frac{e^{|v|x_1 - \phi(u)}}{1 + u^4}.$$

(20.02) follows for $v = 0$. (Actually (20.08) is not needed here but will be used in a later chapter.) Since $h(x)$ vanishes outside of a finite interval, its Fourier transform must be an entire function. This completes the proof of the theorem.

In Theorems XXIII and XXV (and therefore also XXII and XXIV), the conditions on $F(u)$ can be considerably relaxed. In the first place, as $u \to \infty$ the fact that $F(u)$ must be small can be put in a less restrictive form than (18.01) and (18.02) or than (19.25). Secondly, for $u < 0$ not only does $F(u)$ not have to belong to $L(-\infty, 0)$ but $F(u)$ can behave like $e^{|u|^{1/2}}$ for example as $u \to -\infty$. The following theorem is an extension of Theorem XXII along these lines.

THEOREM XXVII. *Let* $e^{-\epsilon|w|}F(u) \epsilon L(-\infty, \infty)$ *for any* $\epsilon > 0$, *and let there exist an even positive function* $\phi(u)$, *increasing with* $| u |$ *and such that*

$$\int_1^{\infty} \frac{\phi(u)}{u^2}\, du < \infty.$$

Let

$$(20.09) \qquad \int_{-\infty}^{0} |F(u)|e^{-\phi(u)}\,du < \infty,$$

and let

$$(20.10) \qquad \int_{1}^{\infty} \frac{du}{u^2} \log \int_{u}^{\infty} |F(v)|e^{-\phi(v)}\,dv = -\infty.$$

If for some a and b, $(b > a)$,

$$(20.11) \qquad \lim_{\epsilon \to 0} \int_{a}^{b} \left| \int_{-\infty}^{\infty} F(u) e^{-\epsilon|u| - iux}\,du \right| dx = 0,$$

then $F(u)$ is zero almost everywhere over $(-\infty, \infty)$.

We observe that (20.11) is a generalization of the requirement that the transform of $F(u)$ vanish over an interval (a, b). Inequality (20.09) is certainly satisfied if $F(u) \in L^p$, $p \geqq 1$, or if $F(u) = O(|u|^A)$ or $O(e^{|u|^{1/2}})$ as $u \to -\infty$. Equation (21.10) is a much weaker requirement than (18.01) and (18.02).

The following theorem generalizes Theorem XXIII (and XXV) in the same way that Theorem XXVII generalizes Theorem XXII.

THEOREM XXVIII. *Let $F(u)$ satisfy the requirements of Theorem XXVII. Let $g(z)$ be analytic in $(a \leqq x \leqq b,\ 0 < y \leqq \gamma)$ and continuous in $(a \leqq x \leqq b,\ 0 \leqq y \leqq \gamma)$. Let there exist an $f(x)$ such that*

$$(20.12) \qquad \lim_{\epsilon \to 0} \int_{a}^{b} \left| f(x) - \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F(u) e^{-\epsilon|u| - iux}\,du \right| dx = 0.$$

If $f(x) = g(x)$ almost everywhere in $(a < \alpha \leqq x \leqq \beta < b)$, then $f(x)$ must equal $g(x)$ almost everywhere in $(a < x < b)$.[5]

Proof of Theorem XXVII. Let $0 < x_0 < \frac{1}{4}(b - a)$ By Theorem XXVI there exists an $h(x)$ which vanishes for $|x| > x_0$ and with Fourier transform $H(u)$ satisfying

$$(20.13) \qquad |H(u)| \leqq \frac{e^{-\phi(u)}}{1 + u^4}$$

where $\phi(u)$ is given in Theorem XXVII. Let

$$F_1(u) = F(u)H(u).$$

Then by (20.13) and (20.09), $F_1(u) \in L(-\infty, 0)$. Again, by (20.10)

$$\int_{2}^{\infty} |F(v)|e^{-\phi(v)}\,dv < \infty.$$

[5] The condition that $g(z)$ be continuous for $(a \leqq x \leqq b,\ 0 \leqq y \leqq \gamma)$ can be replaced by $\int_a^b |g(x + iy) - g(x)|\,dx \to 0$ as $y \to +0$.

Thus $F_1(u) \in L(2, \infty)$. Since $F(u)$ is integrable over $(0, 2)$, we now have $F_1(u) \in L(-\infty, \infty)$. Again by (20.13),

$$\int_u^\infty |F_1(v)| \frac{dv}{v^2} \leqq \int_u^\infty |F(v)H(v)| \, dv \leqq \int_u^\infty |F(v)| e^{-\phi(v)} \, dv.$$

Thus by (20.10)

$$\int_1^\infty \frac{du}{u^2} \log \int_u^\infty |F_1(v)| \frac{dv}{v^2} = -\infty.$$

Thus $F_1(u)$ satisfies the requirements of Theorem XXV.

Let the Fourier transform of $F_1(u)$ be

$$\begin{aligned} f_1(x) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^\infty F_1(u) e^{-iux} \, du \\ &= \lim_{\epsilon \to 0} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^\infty F(u)H(u) e^{-\epsilon|u| - iux} \, du \\ &= \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^\infty F(u) e^{-\epsilon|u| - iux} \, du \int_{-x_0}^{x_0} h(\xi) e^{iu\xi} \, d\xi. \end{aligned}$$

Since $e^{-\epsilon|u|}F(u) \in L(-\infty, \infty)$, it follows that

$$f_1(x) = \lim_{\epsilon \to \infty} \frac{1}{2\pi} \int_{-x_0}^{x_0} h(\xi) \, d\xi \int_{-\infty}^\infty F(u) e^{-\epsilon|u| - iu(x-\xi)} \, du. \tag{20.14}$$

By (20.11) this gives

$$f_1(x) = 0, \qquad a + x_0 < x < b - x_0.$$

Since $F_1(u)$ satisfies Theorem XXV, it follows that $f_1(x) \equiv 0$, $(-\infty < x < \infty)$. Thus its transform, $F_1(u) = H(u)F(u)$, must be zero almost everywhere. Since $H(u)$ is an entire function, it vanishes only at isolated points. Thus $F(u)$ is zero almost everywhere. This completes the proof of Theorem XXVII.

Proof of Theorem XXVIII. Let $0 < x_0 < \frac{1}{4}(\beta - \alpha)$. Since $\phi(2u)$ satisfies the requirements of Theorem XXVI if $\phi(u)$ does, there exists $H(u)$ satisfying

$$|H(u)| \leqq \frac{e^{-\phi(2u)}}{1 + u^4} \tag{20.15}$$

and such that $h(x)$, its transform, vanishes for $|x| \geqq x_0$. Proceeding exactly as in the proof of Theorem XXVII, we obtain (20.14). From (20.14) and (20.12),

$$f_1(x) = \int_{-x_0}^{x_0} h(\xi) f(x - \xi) \, d\xi, \qquad a + x_0 < x < b - x_0. \tag{20.16}$$

We recall that $F_1(u)$, the transform of $f_1(x)$, has been shown to satisfy the requirements of Theorem XXV. Let

$$g_1(z) = \int_{-x_0}^{x_0} h(\xi)g(z - \xi)\, d\xi. \tag{20.17}$$

Then $g_1(z)$ is analytic for $(a + x_0 < x < b - x_0,\ 0 < y < \gamma)$. Thus $g_1(z)$ satisfies the requirements of Theorem XXV. Also by (20.16) and (20.17) and the fact that $f(x) = g(x)$ almost everywhere in (α, β), it follows that $f_1(x) = g_1(x)$, $(\alpha + x_0 < x < \beta - x_0)$. Thus by Theorem XXV,

$$f_1(x) = g_1(x), \qquad a + x_0 < x < b - x_0.$$

That is,

$$\int_{-x_0}^{x_0} h(\xi)\{f(x - \xi) - g(x - \xi)\}\, d\xi = 0, \qquad a + x_0 < x < b - x_0. \tag{20.18}$$

But (20.18) will also hold with $h(\xi)e^{v\xi}$ instead of $h(\xi)$ since the transform of this function is $H(u + v)$, which for any fixed v can be used just as well as $H(u)$ itself in obtaining the above results, providing we can show for any fixed v that

$$H(u + iv) = O(e^{\phi(u)}), \qquad |u| \to \infty. \tag{20.19}$$

But it was just to assure this result that we use $\phi(2u)$ in (20.15) rather than simply $\phi(u)$ itself, since as $|u|$ gets large $2|u| - 2|v|$ soon exceeds $|u|$ in magnitude. Thus for large $|u|$ and fixed v, $\phi(2u + 2v) > \phi(u)$. From this and (20.15), (20.19) follows. Thus instead of (20.18) we now have, for all v,

$$\int_{-x_0}^{x_0} h(\xi)\{f(x - \xi) - g(x - \xi)\}\, e^{iv\xi}\, d\xi = 0, \qquad a + x_0 < x < b - x_0.$$

Thus for almost all ξ, $|\xi| < x_0$,

$$h(\xi)\{f(x - \xi) - g(x - \xi)\} = 0, \qquad a + x_0 < x < b - x_0. \tag{20.20}$$

But $h(0) \neq 0$ and $h(\xi)$ is continuous. Thus there exists a δ, $0 < \delta < x_0$, such that $h(\xi) \neq 0$, $|\xi| \leqq \delta$. Thus (20.20) gives for almost all ξ, $|\xi| \leqq \delta < x_0$,

$$f(x - \xi) = g(x - \xi), \qquad a + x_0 < x < b - x_0.$$

From this it follows at once that $f(x) = g(x)$ for almost all x, $(a + x_0 < x < b - x_0)$. Since x_0 can be chosen arbitrarily small, it follows that $f(x) = g(x)$ for almost all x, $a < x < b$. This completes the proof of Theorem XXVIII.

CHAPTER VI

A DENSITY THEOREM OF PÓLYA

21. The density theorem. In this chapter we shall prove what might be called the classical density theorem although the theorem started as a gap result.

If a sufficient number of the coefficients of a power series are zero, every point on its circle of convergence is a singular point. This was first proved by Hadamard for the power series

$$\sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

where

$$\lambda_{n+1}/\lambda_n \geqq c > 1.$$

An example of such a series is

$$F(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

The points

$$\{k/2^m\}, \qquad 0 < k < 2^n, m = 1, 2, \cdots,$$

are everywhere dense on (0, 1). Let us consider the series $F(z)$ as $z \to e^{2\pi ki/2^m}$ along a radius from $z = 0$. Clearly here $\lim | F(z) | = \infty$. Since this is the case for an everywhere dense set of points on the unit circle, the unit circle is a natural boundary for $F(z)$.

This result has been extended and improved considerably. Let us consider the following series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^{3n}.$$

$f(z)$ must possess at least one singular point on its circle of convergence. Let this point be z_0. Then the points $z_0 e^{2\pi i/3}$ and $z_0 e^{4\pi i/3}$ must also be singular points. Thus if $\lambda_n = 3n$, $f(z)$ will have at least one singular point on every arc on its circle of convergence of circular measure exceeding $2\pi/3$. This result remains true if $\lambda_n = 3n$ is replaced by the much less restrictive condition

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{1}{3}.$$

This follows from the following theorem which is stated for Dirichlet series and therefore includes power series as a special case.

THEOREM XXIX.[1] *Let*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \tag{21.01}$$

where

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D \tag{21.02}$$

and for some $c > 0$

$$\lambda_{n+1} - \lambda_n \geqq c. \tag{21.03}$$

Then on the abscissa of convergence there is at least one singularity in every interval of length exceeding $2\pi D$.

In particular if $D = 0$, the abscissa of convergence is a natural boundary.

Theorems involving a sequence $\{\lambda_n\}$ of density D are very often proved by using the entire function

$$F(z) = \prod_1^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

In other words an important way to make use of the properties of $\{\lambda_n\}$ is by means of the corresponding properties of $F(z)$. For example $n/\lambda_n \to D$ implies that

$$\lim_{|y|\to\infty} \frac{\log F(iy)}{|y|} = \pi D.$$

In certain respects this last result is a much easier one to make use of than $n/\lambda_n \to D$. Thus the proof of Theorem XXIX makes use of the following theorem.

THEOREM XXX.[2] *If* $\{\lambda_n\}$ *satisfies* (21.02) *and* (21.03) *and if*

$$F(z) = \prod_1^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right), \tag{21.04}$$

then as $r \to \infty$, *for* $\epsilon > 0$

$$F(re^{i\theta}) = O(\exp\{\pi D r |\sin\theta| + \epsilon r\}), \tag{21.05}$$

$$\frac{1}{F(re^{i\theta})} = O(\exp\{-\pi D r |\sin\theta| + \epsilon r\}), \qquad |re^{i\theta} - \lambda_n| \geqq \tfrac{1}{4}c, \tag{21.06}$$

[1] Pólya, *Ueber die Existenz unendlich vieler singularer Punkte auf der Konvergenzgeraden gewisser Dirichletscher Reihen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1923, pp. 45–50.

[2] F. Carlson, *Über Potenzreihen mit endlich vielen verschiedenen Koeffizienten*, Mathematische Annalen, vol. 79 (1919), pp. 237–245.

and as[3] $n \to \infty$

(21.07) $$\frac{1}{F'(\lambda_n)} = O(e^{\epsilon\lambda_n}).$$

Theorem XXX will be a corollary of a more general theorem, whose proof will follow that of Theorem XXIX.

Proof of Theorem XXIX. With no loss of generality we can assume that the abscissa of convergence of $f(z)$ is $x = 0$. If there exists an interval of length greater than $2\pi D$ on $x = 0$ on which $f(z)$ has no singularity, we can also assume with no loss of generality that this interval is $-B \leqq y \leqq B$ where

$$B > \pi D.$$

If $f(z)$ contains no singularity on $x = 0$, $|y| \leqq B$, then there exists some $a > 0$ such that $f(z)$ is analytic for $x \geqq -a$, $|y| \leqq B$.

Let[4]

$$H(w) = \int_{-a}^{\infty} f(z)e^{uz}\,dz.$$

Then if $w = u + iv$, $H(u)$ is defined for $u < 0$. By the Cauchy integral theorem, we can change the path of integration to give

$$H(w) = \int_{-a}^{-a+iB} f(z)e^{uz}\,dz + \int_{-a+iB}^{\infty+iB} f(z)e^{wz}\,dz.$$

The path in the second integral is along the line $y = iB$ from $x = -a$ to $x = \infty$. Let $b > 0$. Then

$$H(w) = \int_{-a}^{-a+iB} f(z)e^{uz}\,dz + \int_{-a+iB}^{b+iB} f(z)e^{wz}\,dz + \int_{b+iB}^{\infty+iB} e^{wz}\left(\sum_{1}^{\infty} a_k e^{-\lambda_k z}\right)dz.$$

The condition $\lim n/\lambda_n = D$ causes the Dirichlet series for $f(z)$ to converge absolutely for $x > 0$ just as is the case with power series. Thus

(21.08) $$H(w) = \int_{-a}^{-a+iB} f(z)e^{uz}\,dz + \int_{-a+iB}^{b+iB} f(z)e^{wz}\,dz - \sum_{1}^{\infty} a_k \frac{e^{(w-\lambda_k)(b+iB)}}{w - \lambda_k}.$$

But now $H(w)$ is defined for all w. If $F(w)$ is defined as in Theorem XXX, then by (21.08)

$$G(w) = H(w)F(w)$$

is an entire function, and

(21.09) $$G(\lambda_n) = -a_nF'(\lambda_n).$$

By (21.08) if $w = \rho e^{i\gamma}$, $0 < \gamma < \frac{1}{2}\pi$, then

[3] It is possible to replace (21.03) in Theorems XXIX and XXX by less restrictive conditions. What is essential is that these conditions imply (21.07).

[4] The function $H(w)$ is used by V. Bernstein, *Séries de Dirichlet*, Paris, 1933, p. 111.

$$| H(\rho e^{i\gamma}) | \leqq \exp \{-a\rho \cos \gamma\} \int_{-a}^{-a+iB} | f(z)\, dz |$$

$$+ \exp \{b\rho \cos \gamma - B\rho \sin \gamma\} \int_{-a+iB}^{b+iB} | f(z)\, dz |$$

$$+ \exp \{b\rho \cos \gamma - B\rho \sin \gamma\} \sum_1^\infty \frac{| a_k | e^{-b\lambda_k}}{b \sin \gamma}.$$

By choosing the path of integration as the reflection in the real axis of that used in (21.08), the above inequality holds for $\rho e^{-i\gamma}$. Thus

$$H(\rho e^{\pm i\gamma}) = O(\exp \{-a\rho \cos \gamma\} + \exp \{b\rho \cos \gamma - B\rho \sin \gamma\}).$$

Using this and (21.05),

$$G(\rho e^{\pm i\gamma}) = O(\exp \{-a\rho \cos \gamma + \pi D\rho \sin \gamma + \epsilon\rho\}$$
$$+ \exp \{b\rho \cos \gamma - (B - \pi D)\, \rho \sin \gamma + \epsilon\rho\}).$$

Since $B > \pi D$ we can take

$$b = \tfrac{1}{2}(B - \pi D) \tan \gamma.$$

Then $B - \pi D = 2b \operatorname{ctn} \gamma$ and

$$G(\rho e^{\pm i\gamma}) = O(\exp \{-a\rho \cos \gamma + \pi D\rho \sin \gamma + \epsilon\rho\} + \exp \{-b\rho \cos \gamma + \epsilon\rho\}).$$

Since ϵ is arbitrarily small,

$$G(\rho e^{\pm i\gamma}) = O(\exp \{-\tfrac{1}{2}a\rho \cos \gamma + \pi D\rho \sin \gamma\} + \exp \{-\tfrac{1}{2}b\rho \cos \gamma\}).$$

If $D = 0$ then for $\gamma = \frac{1}{4}\pi$,

$$G(\rho e^{\pm i\gamma}) = O(e^{-\delta\rho \cos \gamma}), \qquad \{\delta = \tfrac{1}{4} \min (a, b)\}.$$

If $D > 0$, let

$$\gamma = \tan^{-1} \frac{a}{4\pi D}.$$

Then $\pi D \sin \gamma = \frac{1}{4}a \cos \gamma$ and

$$G(\rho e^{\pm i\gamma}) = O(\exp \{-\tfrac{1}{4}a\rho \cos \gamma\} + \exp \{-\tfrac{1}{2}b\rho \cos \gamma\}),$$

and again

(21.10) $$G(\rho e^{\pm i\gamma}) = O(e^{-\delta\rho \cos \gamma}).$$

Thus for $\delta > 0$,

(21.11) $$e^{\delta w} G(w) = O(1), \qquad | w | \to \infty, \text{ am } w = \pm \gamma.$$

By (21.08) there exists an A such that for $| w - \lambda_k | \geqq \frac{1}{4}c$, $(k > 0)$,

$$| H(w) | \leqq A e^{A|w|}.$$

Using this and (21.05), then for some other A

$$|G(w)| \leq Ae^{A|w|} \tag{21.12}$$

for $|w - \lambda_k| \geq \frac{1}{4}c$, $(k > 0)$. But $G(w)$ is analytic in the whole plane. Thus the maximum value of $G(w)$ for $|w - \lambda_k| \leq \frac{1}{4}c$ is taken on for $|w - \lambda_k| = \frac{1}{4}c$. Thus (21.12) holds for all w, and for some other A

$$e^{\delta w}G(w) = O(e^{A|w|}), \qquad |w| \to \infty.$$

The above result and (21.11), according to a theorem of Phragmén-Lindelöf (Theorem C), give

$$e^{\delta w}G(w) = O(1), \qquad |w| \to \infty, \; |\operatorname{am} w| \leq \gamma.$$

In particular then

$$G(u) = O(e^{-\delta u}), \qquad u \to \infty. \tag{21.13}$$

But $G(\lambda_n) = -a_nF'(\lambda_n)$ by (21.09), and therefore

$$a_n = O\left(\frac{e^{-\delta\lambda_n}}{|F'(\lambda_n)|}\right).$$

By (21.07) this becomes

$$a_n = O(e^{-\delta\lambda_n+\epsilon\lambda_n}).$$

Since ϵ is arbitrary,

$$a_n = O(e^{-\delta\lambda_n/2}).$$

But this would imply that the abscissa of convergence of $f(z)$ is at $-\frac{1}{2}\delta$ or to the left of $-\frac{1}{2}\delta$, which is contrary to our assumption that it is at zero. This completes the proof of the theorem.

22. A function with zeros having a density. Theorem XXX is obviously contained in

THEOREM XXXI. *Let $\{z_n\}$ be a sequence of complex numbers such that*

$$\lim_{n\to\infty} \frac{n}{z_n} = D \tag{22.01}$$

where D is real, and for some $c > 0$ let

$$|z_n - z_m| \geq c|n - m|. \tag{22.02}$$

If

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{z_n^2}\right), \tag{22.03}$$

then, for any $\epsilon > 0$, as $r \to \infty$

$$F(re^{i\theta}) = O(\exp\{\pi Dr|\sin\theta| + \epsilon r\}), \tag{22.04}$$

(22.05) $$\frac{1}{F(re^{i\theta})} = O(\exp\{-\pi Dr\,|\sin\theta| + \epsilon r\}), \qquad |re^{i\theta} - z_n| \geqq \tfrac{1}{8}c,$$

and, as $n \to \infty$,

(22.06) $$\frac{1}{|F'(z_n)|} = O(e^{\epsilon|z_n|}).$$

Theorem XXXI is required in the next chapter. (22.02) can be considerably weakened but we shall not concern ourselves with this aspect here. Our aim is to use the results like Theorem XXXI to obtain a variety of gap and density theorems rather than carefully to investigate Theorem XXXI and similar results.

Proof of Theorem XXXI. Let us choose an $\epsilon > 0$. For fixed z and ϵ let $\{z_n\}$ be divided into three classes, I, II, and III defined as follows:

(22.07) $$\begin{aligned} &z_n \in \mathrm{I}, \quad |z_n| \leqq (1-\epsilon)|z|, \\ &z_n \in \mathrm{II}, \quad (1-\epsilon)|z| < |z_n| < (1+\epsilon)|z|, \\ &z_n \in \mathrm{III}, \; |z_n| \geqq (1+\epsilon)|z|. \end{aligned}$$

Then

(22.08) $$|F(z)| = \left(\prod_{\mathrm{I}} \prod_{\mathrm{II}} \prod_{\mathrm{III}}\right) \left|1 - \frac{z^2}{z_n^2}\right|.$$

Since $n/z_n \to D$, the number of z_n in II for sufficiently large $|z|$ is less than $2\epsilon|z|(D+\delta)$ for $\delta > 0$. Thus

(22.09) $$\prod_{\mathrm{II}} \left|1 - \frac{z^2}{z_n^2}\right| \leqq \prod_{\mathrm{II}} 3 \leqq 3^{2\epsilon|z|(D+\delta)} < e^{4\epsilon|z|(D+\delta)}.$$

From here on we assume that z is in the right half-plane. For sufficiently large n

(22.10) $$\left|\frac{1}{1 - z^2/z_n^2}\right| = \frac{1}{|1 + z/z_n|\,|1 - z/z_n|} \leqq \frac{1}{|1 - z/z_n|} = \left|\frac{z_n}{z - z_n}\right|.$$

Let a value of n which makes $|z - z_n|$ a minimum be N. Then

$$|z - z_n| \geqq |z - z_N|.$$

Thus

$$|z_N - z_n| \leqq |z - z_N| + |z_n - z| \leqq 2|z - z_n|$$

and therefore

$$\frac{1}{|z - z_n|} \leqq \frac{2}{|z_N - z_n|}.$$

Using this in (22.10) gives

$$\left|\frac{1}{1 - z^2/z_n^2}\right| \leqq \frac{2|z_n|}{|z_N - z_n|}. \tag{22.11}$$

By a prime we denote the omission from a product of the term $n = N$. By (22.11),

$$\prod_{\mathrm{II}}' \frac{1}{|1 - z^2/z_n^2|} \leqq \prod_{\mathrm{II}}' \frac{2|z_n|}{|z_N - z_n|}.$$

If M represents the number of z_n in II, then since $|z_n - z_N| \geqq c|n - N|$,

$$\begin{aligned}\prod_{\mathrm{II}}' \frac{1}{|1 - z^2/z_n^2|} &\leqq 2^M\{(1+\epsilon)|z|\}^M \prod_{\mathrm{II}}' \frac{1}{c|n - N|} \\ &\leqq \left(\frac{3}{c}\right)^M |z|^M \prod_{\mathrm{II}}' \frac{1}{|n - N|}.\end{aligned} \tag{22.12}$$

Clearly at most two terms $|n - N|$ in the product on the right above can be equal. Thus if $[\frac{1}{2}(M - 1)]$ represents the largest integer less than or equal to $\frac{1}{2}(M - 1)$, then

$$\prod_{\mathrm{II}}' |n - N| \geqq ([\tfrac{1}{2}(M - 1)]!)^2.$$

Thus (22.12) becomes

$$\prod_{\mathrm{II}}' \frac{1}{|1 - z^2/z_n^2|} \leqq \left(\frac{3}{c}\right)^M \frac{|z|^M}{([\frac{1}{2}(M - 1)]!)^2}.$$

By Stirling's formula this becomes

$$\begin{aligned}\prod_{\mathrm{II}}' \frac{1}{|1 - z^2/z_n^2|} &\leqq \left(\frac{3}{c}\right)^M |z|^M e^{3M} e^{-M \log M} \\ &\leqq \left(\frac{3}{c}\right)^M e^{3M} \exp\{M \log|z| - M \log M\} \\ &= \left(\frac{3}{c}\right)^M e^{3M} \exp\left\{|z|\left(\frac{M}{|z|} \log \frac{|z|}{M}\right)\right\}.\end{aligned}$$

But $M \leqq 2\epsilon|z|(D + \delta)$ Since $x \log 1/x < x^{1/2}$ for $x > 0$,

$$\frac{M}{|z|} \log \frac{|z|}{M} \leqq \left(\frac{M}{|z|}\right)^{1/2} < \frac{(2\epsilon|z|(D + \delta))^{1/2}}{|z|^{1/2}} = (2\epsilon(D + \delta))^{1/2}.$$

Thus

$$\prod_{\mathrm{II}}' \frac{1}{|1 - z^2/z_n^2|} \leqq \left(\frac{3}{c}\right)^{2\epsilon|z|(D+\delta)} e^{10\epsilon|z|(D+\delta)} e^{|z|(2\epsilon(D+\delta))^{1/2}}. \tag{22.13}$$

Since z lies in the right half-plane,

$$\frac{1}{|1 - z^2/z_N^2|} = \frac{|z_N|}{|z - z_N||1 + z/z_N|} < \frac{2|z_N|}{|z - z_N|}.$$

By the minimum property of z_N, for large $|z|$, $|z_N| < 2|z|$. Therefore

$$\frac{1}{|1 - z^2/z_N^2|} \leqq \frac{4|z|}{|z - z_N|}.$$

Also

$$\frac{4|z|}{|z - z_N|} \geqq \frac{4|z|}{|z| + |z_N|} \geqq \frac{4|z|}{|z| + 2|z|} > 1.$$

Thus regardless of whether or not $z_N \in \mathrm{II}$,

$$\prod_{\mathrm{II}} \frac{1}{|1 - z^2/z_n^2|} \leqq \frac{4|z|}{|z - z_N|} \prod_{\mathrm{II}}{}' \frac{1}{|1 - z^2/z_n^2|}.$$

Using (23.13) it follows for large $|z|$ that

$$\prod_{\mathrm{II}} \frac{1}{|1 - z^2/z_n^2|} \leqq \left(\frac{3}{c}\right)^{2\epsilon|z|(D+\delta)} \frac{e^{10\epsilon|z|(D+\delta)} e^{|z|(3\epsilon(D+\delta))^{1/2}}}{|z - z_N|}.$$

It follows from this result and (22.09) that there exists an A independent of $|z|$, δ, and ϵ such that for large $|z|$

$$(22.14) \qquad |z - z_N| e^{-A|z|\epsilon^{1/2}(D+\delta)^{1/2}} \leqq \prod_{\mathrm{II}} \left|1 - \frac{z^2}{z_n^2}\right| \leqq e^{A|z|\epsilon^{1/2}(D+\delta)^{1/2}}.$$

If $D = 0$ let $\epsilon = \frac{1}{2}$. Then

$$(22.15) \qquad |z - z_N| e^{-A|z|\delta^{1/2}} \leqq \prod_{\mathrm{II}} \left|1 - \frac{z^2}{z_n^2}\right| \leqq e^{A|z|\delta^{1/2}}.$$

If $D > 0$ let $\delta = 1$. Then if $B = A(D + 1)^{1/2}$

$$(22.16) \qquad |z - z_N| e^{-B|z|\epsilon^{1/2}} \leqq \prod_{\mathrm{II}} \left|1 - \frac{z^2}{z_n^2}\right| \leqq e^{B|z|\epsilon^{1/2}}.$$

Until otherwise stated we shall assume $D > 0$ in what follows. For sufficiently large N_0, since $n/z_n \to D$,

$$(22.17) \qquad \left|\frac{D^2 z_n^2 - n^2}{z_n^2}\right| \leqq \tfrac{1}{8} D^2 \epsilon^2, \qquad n \geqq N_0.$$

Let $z_n \in \mathrm{I}$ or III, $(n \geqq N_0)$, and $|z_n| \leqq 2|z|$. Clearly

$$|n^2 - z^2 D^2| \geqq |D^2 z_n^2 - z^2 D^2| - |n^2 - D^2 z_n^2|.$$

Since $z_n \in \mathrm{I}$ or III, $||z_n| - |z|| \geqq \epsilon|z|$. Using this and (22.17) the above inequality becomes

$$\begin{aligned} |n^2 - z^2 D^2| &\geqq D^2 \epsilon |z| \, |z_n + z| - \tfrac{1}{8} D^2 \epsilon^2 |z_n|^2 \\ &\geqq D^2 \epsilon |z| \, |\tfrac{1}{2} z| - \tfrac{1}{2} D^2 \epsilon^2 |z|^2 = \tfrac{1}{2} D^2 \epsilon |z|^2 (1 - \epsilon). \end{aligned}$$

For sufficiently small ϵ this becomes

$$\left|\frac{n^2 - D^2z^2}{z^2}\right| \geqq \tfrac{1}{4}D^2\epsilon. \tag{22.18}$$

Using (22.17) and (22.18),

$$1 - \epsilon \leqq \left|1 + \frac{z^2(z_n^2 D^2 - n^2)}{z_n^2(n^2 - D^2z^2)}\right| \leqq 1 + \epsilon, \qquad n \geqq N_0, \tag{22.19}$$

for $|z_n| \leqq 2|z|$ and $z_n \in$ I or III. Since

$$1 + \frac{z^2(z_n^2 D^2 - n^2)}{z_n^2(n^2 - z^2 D^2)} = \frac{1 - z^2/z_n^2}{1 - z^2D^2/n^2}, \tag{22.20}$$

(22.19) becomes

$$1 - \epsilon \leqq \left|\frac{1 - z^2/z_n^2}{1 - z^2D^2/n^2}\right| \leqq 1 + \epsilon. \tag{22.21}$$

Clearly for large $|z|$ there exists a positive constant C_1, such that

$$\frac{1}{C_1} \leqq \prod_1^{N_0}\left|\frac{1 - z^2/z_n^2}{1 - z^2D^2/n^2}\right| \leqq C_1.$$

The last two results give, since the number of z_n in I is less than $2|z|D$ for large $|z|$,

$$\begin{aligned}\frac{1}{C_1}(1-\epsilon)^{2|z|D}\prod_{\mathrm{I}}\left|1 - \frac{z^2D^2}{n^2}\right| &\leqq \prod_{\mathrm{I}}\left|1 - \frac{z^2}{z_n^2}\right| \\ &\leqq C_1(1+\epsilon)^{2|z|D}\prod_{\mathrm{I}}\left|1 - \frac{z^2D^2}{n^2}\right|.\end{aligned} \tag{22.22}$$

For $|z_n| \geqq 2|z|$, by (22.17)

$$1 - \frac{1}{8}\frac{D^2\epsilon^2|z|^2}{n^2 - |z|^2D^2} \leqq \left|1 + \frac{z^2(z_n^2 D^2 - n^2)}{z_n^2(n^2 - z^2D^2)}\right| \leqq 1 + \frac{1}{8}\frac{D^2\epsilon^2|z|^2}{n^2 - |z|^2D^2}.$$

Since $z_n \sim n/D$, for large $|z|$ this becomes

$$1 - \frac{\epsilon|z|^2D^2}{n^2} \leqq \left|1 + \frac{z^2(z_n^2 D^2 - n^2)}{z_n^2(n^2 - z^2D^2)}\right| \leqq 1 + \frac{\epsilon|z|^2D^2}{n^2}.$$

Using this result with (22.20) and (22.21),

$$\begin{aligned}(1-\epsilon)^{2|z|D}\prod_{|z_n|\geqq 2|z|}\left(1 - \frac{\epsilon|z|^2D^2}{n^2}\right) &\leqq \prod_{\mathrm{III}}\left|\frac{1 - z^2/z_n^2}{1 - D^2z^2/n^2}\right| \\ &\leqq (1+\epsilon)^{2|z|D}\prod_{|z_n|\geqq 2|z|}\left(1 + \frac{\epsilon|z|^2D^2}{n^2}\right).\end{aligned} \tag{22.23}$$

Since for small x, $1 + x < e^x$ and $1 - x > e^{-2x}$, it follows that

$$\prod_{|z_n|\geqq 2|z|}\left(1 + \frac{\epsilon|z|^2D^2}{n^2}\right) \leqq \exp\left\{\epsilon|z|^2D^2\sum_{n>|z|D}\frac{1}{n^2}\right\} \leqq e^{2\epsilon|z|D}$$

and

$$\prod_{|z_n|\geqq 2|z|}\left(1-\frac{\epsilon|z|^2D^2}{n^2}\right)\geqq \exp\left\{-2\epsilon|z|^2D^2\sum_{n>|z|D}\frac{1}{n^2}\right\}\geqq e^{-4\epsilon D|z|}.$$

Thus (22.23) becomes

$$e^{-8\epsilon D|z|}\leqq \prod_{\mathrm{III}}\left|\frac{1-z^2/z_n^2}{1-z^2D^2/n^2}\right|\leqq e^{4\epsilon D|z|}.$$

Combining this result with (22.22), (22.16), and (22.08),

$$(22.24)\qquad \begin{aligned}&\frac{|z-z_N|}{C_1}e^{-B|z|\epsilon^{1/2}-12\epsilon|z|D}\prod_{\mathrm{I,III}}\left|1-\frac{z^2D^2}{n^2}\right|\\ &\qquad\leqq |F(z)|\leqq C_1e^{B|z|\epsilon^{1/2}+6\epsilon D|z|}\prod_{\mathrm{I,III}}\left|1-\frac{z^2D^2}{n^2}\right|.\end{aligned}$$

From the product formula for $\sin\pi z$,

$$(22.25)\qquad \prod_{\mathrm{I,III}}\left|1-\frac{z^2D^2}{n^2}\right|=\left|\frac{\sin\pi Dz}{\pi Dz}\right|\prod_{\mathrm{II}}\frac{1}{|1-z^2D^2/n^2|}.$$

Let N_1 be a value of n such that $|Dz-n|$ is a minimum for $n=N_1$. Then proceeding just as in getting (22.16),

$$(22.26)\qquad |zD-N_1|e^{-B|z|\epsilon^{1/2}}\leqq\prod_{\mathrm{II}}\left|1-\frac{z^2D^2}{n^2}\right|\leqq e^{B|z|\epsilon^{1/2}}.$$

Using this result in (22.25) gives for large $|z|$

$$|\sin\pi Dz|e^{-2B|z|\epsilon^{1/2}}\leqq\prod_{\mathrm{I,III}}\left|1-\frac{z^2D^2}{n^2}\right|\leqq\left|\frac{\sin\pi Dz}{zD-N_1}\right|e^{2B|z|\epsilon^{1/2}}.$$

The above result in (22.24) gives

$$(22.27)\qquad \begin{aligned}&\frac{1}{C_1}|z-z_N|\,|\sin\pi Dz|e^{-3B|z|\epsilon^{1/2}-12\epsilon D|z|}\\ &\qquad\leqq|F(z)|\leqq C_1\left|\frac{\sin\pi Dz}{zD-N_1}\right|e^{3B|z|\epsilon^{1/2}+6\epsilon D|z|}.\end{aligned}$$

For large $|z|$ if $z=re^{i\theta}$,

$$\left|\frac{\sin\pi Dz}{zD-N_1}\right|\leqq\exp\{\pi Dr|\sin\theta|\}.$$

Thus (22.27) gives

$$|F(z)|\leqq C_1\exp\{3Br\epsilon^{1/2}+6\epsilon Dr+\pi Dr|\sin\theta|\}.$$

Redefining ϵ this gives

$$F(z)=O(\exp\{\pi Dr|\sin\theta|+\epsilon r\})$$

which is (22.04).

Again from (22.27)

$$\left|\frac{F(z)}{z - z_N}\right| \geqq \frac{1}{C_1} \exp \{-3B|z|\epsilon^{1/2} - 12\epsilon D|z|\} \, |\sin \pi Dz|. \tag{22.28}$$

In exactly the same way as (22.28) is obtained,

$$\left|\frac{F(z)}{z - z_N}\right| \geqq \frac{1}{C_1} \exp \{-3B|z|\epsilon^{1/2} - 12\epsilon D|z|\} \, |\cos \pi Dz|$$

can be obtained. Thus

$$\left|\frac{F(z)}{z - z_N}\right| \geqq \frac{1}{C_1} \exp \{-3B|z|\epsilon^{1/2} - 12\epsilon D|z|\} \max (|\cos \pi Dz|, |\sin \pi Dz|).$$

Redefining ϵ it follows from this for large $|z|$ that

$$\left|\frac{F(z)}{z - z_N}\right| \geqq \exp \{\pi Dr|\sin \theta| - \epsilon r\}. \tag{22.29}$$

Letting $z \to z_k$, z_N must eventually become z_k. Thus (22.29) gives

$$|F'(z_k)| \geqq e^{-\epsilon|z_k|},$$

which is (22.06). From (22.29) it also follows at once that

$$\frac{1}{F(z)} = O(\exp \{-\pi Dr|\sin \theta| + \epsilon r\}), \qquad |z - z_k| \geqq \tfrac{1}{8}c, \quad k = 1, 2, \cdots,$$

which is (22.05). The results proved so far all hold for z in the right half-plane. But since $F(z)$ is even they must hold for all z. Thus for $D > 0$ we have demonstrated Theorem XXXI.

If $D = 0$ we already have (22.15) which states

$$|z - z_N| e^{-A|z|\delta^{1/2}} \leqq \prod_{\mathrm{II}} \left|1 - \frac{z^2}{z_n^2}\right| \leqq e^{A|z|\delta^{1/2}} \tag{22.30}$$

where $z_n \in \mathrm{II}$ if $\frac{1}{2}|z| < |z_n| < \frac{3}{2}|z|$. In I, $|z_n| \leqq \frac{1}{2}|z|$. Thus

$$1 \leqq \prod_{\mathrm{I}} \left|1 - \frac{z^2}{z_n^2}\right| \leqq \prod_{\mathrm{I}} 2\left|\frac{z^2}{z_n^2}\right|. \tag{22.31}$$

For large $|z_n|$, if $D = 0$, $|z_n| > n$. If K is the number of $z_n \in \mathrm{I}$, then

$$\prod_{\mathrm{I}} 2\left|\frac{z^2}{z_n^2}\right| = O\left(2^K \frac{|z|^{2K}}{(K!)^2}\right).$$

Using Stirling's formula for $K!$, this becomes

$$\prod_{\mathrm{I}} 2\left|\frac{z^2}{z_n^2}\right| = O\left(2^K \left|\frac{z}{K}\right|^{2K} e^{4K}\right) = O(\exp \{5K + 2K \log (|z|/K)\}). \tag{22.32}$$

But for large $|z|$ since $D = 0$, $k < \epsilon|z|$ for any $\epsilon > 0$. But since $x \log 1/x < x^{1/2}$,

$$\frac{K}{|z|}\log\frac{|z|}{K} \leqq \left(\frac{K}{|z|}\right)^{1/2} < \left(\frac{\epsilon|z|}{|z|}\right)^{1/2} = \epsilon^{1/2}.$$

Using this and $K < \epsilon |z|$ in (22.32),

$$\prod_{\mathrm{I}} 2\left|\frac{z^2}{z_n^2}\right| = O(e^{5\epsilon|z|+|z|\epsilon^{1/2}}).$$

Thus (22.31) becomes

$$1 \leqq \prod_{\mathrm{I}}\left|1 - \frac{z^2}{z_n^2}\right| = O(e^{5\epsilon|z|+|z|\epsilon^{1/2}}). \tag{22.33}$$

In III, $|z_n| \geqq \frac{3}{2}|z|$. Thus

$$\prod_{\mathrm{III}}\left(1 - \frac{r^2}{r_n^2}\right) \leqq \prod_{\mathrm{III}}\left|1 - \frac{z^2}{z_n^2}\right| \leqq \prod_{\mathrm{III}}\left(1 + \frac{r^2}{r_n^2}\right). \tag{22.34}$$

But

$$\prod_{\mathrm{III}}\left(1 + \frac{r^2}{r_n^2}\right) \leqq \exp\left\{r^2 \sum_{\mathrm{III}} \frac{1}{r_n^2}\right\}.$$

Let $R(u)$ be the number of $r_n < u$. Then the above inequality becomes

$$\prod_{\mathrm{III}}\left(1 - \frac{r^2}{r_n^2}\right) \leqq \exp\left\{r^2 \int_{3r/2}^{\infty} \frac{dR(u)}{u^2}\right\}$$
$$\leqq \exp\left\{r^2 \int_{3r/2}^{\infty} \frac{2R(u)}{u^3}\,du\right\} \leqq \exp\left\{r \max_{u \geq 3r/2} \frac{R(u)}{u}\right\} \leqq e^{\epsilon r}$$

for large r since $R(u)/u \to 0$ as $u \to \infty$. In much the same way

$$\prod_{\mathrm{III}}\left(1 - \frac{r^2}{r_n^2}\right) \geqq e^{-2\epsilon r}.$$

Thus (22.34) becomes

$$e^{-2\epsilon|z|} \leqq \prod_{\mathrm{III}}\left|1 - \frac{z^2}{z_n^2}\right| \leqq e^{\epsilon|z|}.$$

Combining this with (22.30) and (22.33) gives

$$|z - z_N|\, e^{A|z|\delta^{1/2}\ 2\epsilon|z|} \leqq |F(z)|$$
$$= O(\exp\{A|z|\delta^{1/2} + 5\epsilon|z| + |z|\epsilon^{1/2} + \epsilon|z|\}).$$

Redefining ϵ in terms of ϵ and δ above, this gives

$$|z - z_N|\, e^{-\epsilon|z|} \leqq |F(z)| \leqq e^{\epsilon|z|}.$$

The conclusions of the theorem now follow easily for $D = 0$.

CHAPTER VII

DETERMINATION OF THE RATE OF GROWTH OF ANALYTIC FUNCTIONS FROM THEIR GROWTH ON SEQUENCES OF POINTS[1]

23. Theorems of V. Bernstein. In this chapter we shall prove a set of theorems which, roughly stated, show that the rate of growth of an analytic function along a line can be determined by its rate of growth along a sufficiently dense sequence of points on the line. We shall deal exclusively with functions which are analytic and of order one in a sector. (Functions of other rates of growth can be transformed by mapping so as to fall under the scope of our theorems.)

The first theorem we shall prove here is the following.

THEOREM XXXII.[2] *Let $\phi(z)$ be analytic in some sector* $|\operatorname{am} z| \leqq \alpha$. *Suppose*

$$\limsup_{r\to\infty} \frac{\log|\phi(re^{i\theta})|}{r} \leqq a\cos\theta + b|\sin\theta|, \qquad |\theta| \leqq \alpha. \tag{23.01}$$

Let $\{z_n\}$ be a sequence of complex numbers such that

$$\lim_{n\to\infty} \frac{n}{z_n} = D, \tag{23.02}$$

where D is real, and such that for some $d > 0$

$$|z_n - z_m| \geqq |n - m|d. \tag{23.03}$$

If

$$\pi D > b, \tag{23.04}$$

then

$$\limsup_{n\to\infty} \frac{\log|\phi(z_n)|}{|z_n|} = \limsup_{r\to\infty} \frac{\log|\phi(r)|}{r}. \tag{23.05}$$

A special case of Theorem XXXII is the following result.

THEOREM XXXIII. *Let $\phi(z)$ be analytic and of finite exponential type*[3] *in the*

[1] Cf. Levinson, *On the growth of analytic functions*, Transactions of the American Mathematical Society, vol 43 (1938), p 240

[2] The theorems in §23 are due to V Bernstein, *Séries de Dirichlet*, Paris, 1933, chap. 9. However Bernstein's proofs involved rather deep theorems in Dirichlet series which are entirely dispensed with here and replaced by methods of ordinary function theory. Also instead of the complex sequence $\{z_n\}$, used here, Bernstein's proofs are restricted to a real sequence.

[3] That is, $\phi(z) = O(e^{C|z|})$ for some C in the region under consideration.

half-plane $|\operatorname{am} z| \leqq \frac{1}{2}\pi$. *Let*

$$\limsup_{|y|\to\infty} \frac{\log|\phi(iy)|}{y} = \pi L. \tag{23.06}$$

If $\{z_n\}$ *satisfies* (23.02) *and* (23.03), *then*

$$D > L \tag{23.07}$$

implies (23.05).

It is easy to see by considering $\sin \pi z$ at the points $z_n = n$ that (23.07) is critical, for in this case $D = L = 1$ and it is clear that (23.05) is not true. Nevertheless we shall show in §24 that (23.07) can be replaced by a much weaker condition.

We now turn to the proof of these theorems.

Proof of Theorem XXXII. We observe that there is no loss of generality in taking $\alpha < \frac{1}{2}\pi$. Clearly, if (23.05) does not hold there exists a c such that

$$\limsup_{n\to\infty} \frac{\log|\phi(z_n)|}{|z_n|} < c < a = \limsup_{x\to\infty} \frac{\log|\phi(x)|}{x}. \tag{23.08}$$

Let

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{z_n^2}\right).$$

Then it follows from (23.08) and (22.06) of Theorem XXXI that the series

$$g(z) = \sum_1^\infty \frac{\phi(z_n)e^{-cz_n}}{F'(z_n)(z - z_n)} e^{cz} F(z) \tag{23.09}$$

converges and represents an entire function. Clearly

$$g(z_n) = \phi(z_n).$$

Therefore

$$\psi(z) = \frac{\phi(z) - g(z)}{F(z)} \tag{23.10}$$

is analytic for $|\operatorname{am} z| \leqq \alpha$. Using (22.05), (23.01), and (23.09), it follows from

$$\psi(z) = \frac{\phi(z)}{F(z)} - \sum_1^\infty \frac{\phi(z_n)e^{-cz_n}}{F'(z_n)(z - z_n)} e^{cz} \tag{23.11}$$

that

$$\psi(z) = O(e^{(|c|+|a|+|b|)|z|}) \tag{23.12}$$

for $|z - z_n| \geqq \frac{1}{4}d$. But $\psi(z)$ is analytic for $|\operatorname{am} z| \leqq \alpha$. Since (23.12) is true on the circles $|z - z_n| \geqq \frac{1}{4}d$, it must be true inside, since an analytic function takes its maximum value in a domain on the boundary. Thus (23.12) holds in the whole sector.

From (23.11), (23.01) and (22.05) it follows that

$$\psi(re^{\pm i\alpha}) = O(\exp\{r(a\cos\alpha + b\sin\alpha - \pi D\sin\alpha + \epsilon)\} + \exp\{cr\cos\alpha\})$$

for any $\epsilon > 0$. Or setting $(\pi D - b)\tan\alpha = \gamma$,

(23.13) $$\psi(re^{\pm i\alpha}) = O(e^{r\cos\alpha(a-\gamma+\epsilon\sec\alpha)} + e^{cr\cos\alpha}).$$

Since $\pi D > b$, $\gamma > 0$. If we take $\epsilon < \frac{1}{2}\gamma\cos\alpha$, then (23.13) becomes

$$\psi(re^{\pm i\alpha}) = O(e^{pr\cos\alpha}), \qquad p = \max(a - \tfrac{1}{2}\gamma, c).$$

In other words $\psi(z)e^{-pz}$ is bounded for $\operatorname{am} z = \pm\alpha$. By the Phragmén-Lindelöf theorem, Theorem C, this and (23.12) imply that $\psi(z)e^{-pz}$ is bounded in the whole sector $(-\alpha, \alpha)$. Thus, in particular,

(23.14) $$\psi(x) = O(e^{px}), \qquad p = \max(a - \tfrac{1}{2}\gamma, c).$$

But from (23.10)

$$\phi(x) = \psi(x)F(x) + g(x).$$

Using (23.11), (22.04), and (23.09), this gives

$$\limsup_{x\to\infty} \frac{\log|\phi(x)|}{x} \leqq \max(a - \tfrac{1}{2}\gamma, c).$$

This contradicts our initial assumption (23.08) and proves the theorem.

Proof of Theorem XXXIII. Let us set

$$\limsup_{x\to\infty} \frac{\log|\phi(x)|}{x} = a.$$

Then by (23.06) and the Phragmén-Lindelof theorem, Theorem C″,

$$\limsup_{r\to\infty} \frac{\log|\phi(re^{i\theta})|}{r} \leqq a\cos\theta + \pi L|\sin\theta|, \qquad |\theta| \leqq \tfrac{1}{2}\pi,$$

in the right half-plane. Theorem XXXIII now follows from Theorem XXXII.

THEOREM XXXIV. *Let $\Phi(z)$ be an analytic function in the right half-plane, $|\operatorname{am} z| \leqq \frac{1}{2}\pi$, such that*

(23.15) $$\Phi(re^{i\theta}) = O(\exp\{(a\log r\cos\theta + \pi b|\sin\theta| + \epsilon)r\}), \qquad |\theta| \leqq \tfrac{1}{2}\pi,$$

where $a \geqq 0$, $b \geqq -\frac{1}{2}a$, and ϵ is an arbitrary positive quantity. If $\{z_n\}$ satisfies (23.02) *and* (23.03) *and if*

(23.16) $$D > b + \tfrac{1}{2}a,$$

then

(23.17) $$\limsup_{n\to\infty} \frac{\log|\Phi(z_n)|}{|z_n|} \leqq \pi p$$

implies

(23.18) $$\Phi(re^{i\theta}) = O(\exp\{\pi(p\cos\theta + b\,|\sin\theta| + \epsilon)r\}), \qquad |\theta| \leqq \tfrac{1}{2}\pi.$$

To show that (23.16) is critical we consider $\Phi(z) = \Gamma(1 + z)\sin\frac{1}{2}\pi z$ for $z_n = 2n$. Then $D = \frac{1}{2}$, and by Stirling's formula for $\Gamma(1 + z)$ for complex z, Theorem F,

$$\Phi(re^{i\theta}) = O(\exp\{r\log r\cos\theta + \epsilon r\}), \qquad |\theta| \leqq \tfrac{1}{2}\pi.$$

Thus $b = 0$, $a = 1$ and therefore here $D = b + \frac{1}{2}a$. But here $p = -\infty$, and therefore the theorem does not hold.

Proof of Theorem XXXIV. We shall assume that $a > 0$, for if $a = 0$ we have Theorem XXXIII. Let

(23.19) $$\phi(z) = \frac{\Phi(z)}{\Gamma(1 + az)}.$$

From Stirling's formula it follows easily that for $|\theta| \leqq \frac{1}{2}\pi$ and large r

$$\log|\Gamma(1 + are^{i\theta})| = ar\log r\cos\theta - ar\theta\sin\theta - ar\cos\theta + \tfrac{1}{2}\log r + O(1).$$

Thus from (23.15)

(23.20) $$\phi(re^{i\theta}) = O(\exp\{(\pi b\,|\sin\theta| + a\theta\sin\theta + a\cos\theta + \epsilon)r\}), \ |\theta| \leqq \tfrac{1}{2}\pi,$$

and from (23.17)

(23.21) $$\phi(z_n) = O(\exp\{(-a\log|z_n|\cos\theta_n + \pi p + a + \epsilon)\,|z_n|\}).$$

From (23.20)

(23.22) $$\limsup_{r\to\infty}\frac{\log|\phi(re^{i\theta})|}{r} \leqq \pi(b + \tfrac{1}{2}a)r\,|\sin\theta| + ar\cos\theta,$$

and from (23.21)

(23.23) $$\limsup_{n\to\infty}\frac{\log|\phi(z_n)|}{|z_n|} = -\infty.$$

Using $D > b + \frac{1}{2}a$ and the above two results in Theorem XXXIII gives

(23.24) $$\limsup_{x\to\infty}\frac{\log|\phi(x)|}{x} = -\infty.$$

We define

(23.25) $$g(z) = \sum_1^\infty \frac{\phi(z_n)e^{Az_n}}{F'(z_n)(z - z_n)}\,e^{-Az}F(z)$$

where A is any real number. That $g(z)$ exists follows from (23.23). We also consider

$$\psi(z) = \frac{\phi(z) - g(z)}{F(z)}.$$

Since $g(z_n) = \phi(z_n)$, $\psi(z)$ is analytic for $|\operatorname{am} z| \leqq \frac{1}{2}\pi$. From (22.05), (23.22),

and (23.25) it follows as in the proof of Theorem XXXII that

(23.26) $$\psi(z) = O(e^{10(a+|b|+D)r}).$$

Along the imaginary axis (assuming $\Re z_n > 1$ as we may with no restriction)

$$|\psi(iy)| \leqq \max\left|\frac{\phi(iy)}{F(iy)}\right| + \sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|.$$

From (22.05), (23.22), and $D > b + \frac{1}{2}a$ it follows that there exists an $M_1 > 0$ independent of A such that

$$\left|\frac{\phi(iy)}{F(iy)}\right| \leqq M_1.$$

Thus

(23.27) $$|\psi(iy)| \leqq M_1 + \sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|.$$

Since by (23.25)

$$g(z) = O(\exp\{-(A - \epsilon)r\cos\theta + (\pi D + \epsilon)r\sin\theta\}), \quad r \to \infty, |\theta| \leqq \tfrac{1}{2}\pi,$$

and by (23.24)

$$\phi(z) = O(e^{Ar\cos\theta}), \qquad r \to \infty, |\theta| < \tfrac{1}{2}\pi,$$

it follows easily from the definition of $\psi(z)$ that

(23.28) $$\psi(x) = O(e^{-(A-\epsilon)x}), \qquad x \to \infty.$$

Thus using (23.26), (23.27), and (23.28)

$$\psi(z) = O(e^{(A-\epsilon)r\cos\theta})$$

by Theorem C′ of Phragmén-Lindelof. Thus $\psi(z)e^{(A-\epsilon)z}$ is bounded in the right half-plane. It follows from (23.27), applying the Theorem C′ of Phragmén-Lindelöf, that

$$|e^{(A-\epsilon)z}\psi(z)| \leqq M_1 + \sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|, \qquad |\operatorname{am} z| \leqq \tfrac{1}{2}\pi.$$

Since the right side is independent of ϵ, ϵ can be taken as zero and thus

(23.29) $$|\psi(x)| \leqq e^{-Ax}\left(M_1 + \sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|\right).$$

Using M_2 and M_3 to represent constants independent of A, it follows from (23.25) that

$$|g(x)| \leqq M_2 e^{-(A-\epsilon)x}\sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|.$$

Since

$$\phi(x) = g(x) + \psi(x)F(x)$$

it follows from the above two inequalities and $F(x) = O(e^{\epsilon x})$ that

$$|\phi(x)| \leqq M_3 e^{-(A-\epsilon)x}\left(1 + \sum_1^\infty \left|\frac{\phi(z_n)e^{Az_n}}{F'(z_n)}\right|\right), \qquad x > 0.$$

Or setting $A = a \log x$, this becomes

$$(23.30) \qquad |\phi(x)| \leqq M_3 e^{-ax \log x + \epsilon x}\left(1 + \sum_1^\infty \left|\frac{\phi(z_n)e^{az_n \log x}}{F'(z_n)}\right|\right), \qquad x > 0.$$

From (23.21) and (22.06),

$$(23.31) \qquad \begin{aligned} &\sum_1^\infty \left|\frac{\phi(z_n)e^{az_n \log x}}{F'(z_n)}\right| \\ &\quad = O\left(\sum_1^\infty \exp\{-a|z_n| \log |z_n/x| \cos\theta_n + (\pi p + a + \epsilon)|z_n|\}\right). \end{aligned}$$

But for $x > 0$,

$$\max_{0 \leqq u < \infty} (e^{-\lambda u \log u/x}) = e^{\lambda x/e}.$$

If $B > 1$ is so large that $\log B > 2(\pi p + a + \epsilon)/a$, then (23.21) becomes

$$\begin{aligned} \sum_1^\infty \left|\frac{\phi(z_n)e^{az_n \log x}}{F'(z_n)}\right| &= O\Big(\exp\{(\pi p + a + \epsilon)Bx + ax/e\} \sum_{|z_n| \leqq Bx} 1 \\ &\quad + \sum_{|z_n| > Bx} \exp\{-a|z_n| \log B \cos\theta_n + (\pi p + a + \epsilon)|z_n|\}\Big) \\ &= O(x \exp\{(\pi p + a + \epsilon)Bx + ax/e\} \\ &\quad + \sum_{|z_n| > Bx} \exp\{-|z_n|(2\cos\theta_n - 1)(\pi p + a + \epsilon)\}) \\ &= O(\exp\{(\pi p + 2a + \epsilon)Bx\}). \end{aligned}$$

Using this in (23 30),

$$\phi(x) = O(\exp\{-ax \log x + Cx\})$$

where $C = (\pi p + 2a + \epsilon)B + \epsilon$. Using this in (23.19),

$$\Phi(x) = O(\Gamma(1 + ax) \exp\{-ax \log x + Cx\}).$$

Using Stirling's formula this becomes

$$\Phi(x) = O(e^{Cx}).$$

Using (23.15)

$$\Phi(iy) = O(e^{(\pi b + \epsilon)|y|}).$$

Using these last two results and (23.15) in Theorem C′ of Phragmén-Lindelöf, it follows that

(23.32) $$\limsup_{r\to\infty} \frac{\log |\Phi(re^{i\theta})|}{r} \leqq \pi b \,|\sin\theta| + C\cos\theta.$$

Theorem XXXIV now follows from Theorem XXXIII.

THEOREM XXXV. *If in Theorem* XXXIV, (23.17) *is replaced by*

(23.33) $$\Phi(z_n) = O(e^{kz_n \log |z_n|})$$

where $k > 0$, *then* (23.18) *is replaced by*

(23.34) $$\Phi(re^{i\theta}) = O(\exp\{(-k\log r\cos\theta + \pi b\,|\sin\theta| + \epsilon)r\}),\quad (|\theta| \leqq \tfrac{1}{2}\pi).$$

Proof of Theorem XXXV. This proof is very much like the preceding one. Here again we set

$$\phi(z) = \frac{\Phi(z)}{\Gamma(1 + az)}$$

and proceed with the single change of

$$\phi(z_n) = O(\exp\{-(a + k)\,|z_n|\log|z_n|\cos\theta_n + (a + \epsilon)\,|z_n|\})$$

in place of (23.21) until we reach (23.30) which, because we now define A as $(a + k)\log x$, becomes

$$|\phi(x)| \leqq M_3 \exp\{-(a + k)x\log x + \epsilon x\}\left(1 + \sum_1^\infty \left|\frac{\phi(z_n)\exp\{(a + k)z_n\log x\}}{F'(z_n)}\right|\right),$$
$$x > 0.$$

Continuing from here as in the preceding proof leads to

$$\Phi(x) = O(\exp\{-kx\log x + Cx\})$$

instead of to the $\Phi(x) = O(e^{Cx})$ of Theorem XXXIV. This leads instead of to (23.32) to the fact that $\Phi(z)\Gamma(1 + kz)$ is of exponential type. Theorem XXXIII can now be applied to $\Phi(z)\Gamma(1 + kz)$ to give Theorem XXXV.

THEOREM XXXVI. *If* $\Phi(z)$ *satisfies the requirements* of Theorem XXXIV *with* (23.17) *replaced by*

(23.35) $$\Phi(z_n) = O(e^{-kz_n\log|z_n|}),$$

and if

(23.36) $$k > 2b,$$

then

(23.37) $$\Phi(z) \equiv 0.$$

Proof of Theorem XXXVI. $\Phi(z)$ clearly satisfies the requirements of Theorem XXXV. By applying Carleman's theorem, Theorem B, to $\Phi(z)$ we have, assuming $\Phi(z)$ is not identically zero,

$$-A \leqq \frac{1}{2\pi}\int_1^R \log^+ |\Phi(iy)\Phi(-iy)| \left(\frac{1}{y^2} - \frac{1}{R^2}\right) dy + \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \log^+ |\Phi(Re^{i\theta})| \cos\theta \, d\theta$$

where A is some constant. Using (23.34) and replacing A by another constant A_1, we get

$$-A_1 \leqq (b+\epsilon)\int_1^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right) y\,dy - \frac{k \log R}{\pi}\int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$

or

$$-A_1 \leqq (b+\epsilon)\log R - \tfrac{1}{2}k \log R.$$

Letting $R \to \infty$, it follows that

$$2(b+\epsilon) \geqq k.$$

Since ϵ is arbitrary, it follows that

$$2b \geqq k.$$

But this contradicts (23.26). Thus $\Phi(z) \equiv 0$.

24. A sharper set of theorems. In this section we shall prove sharper theorems than those of §23. An example of the results of this section is given by the following theorem.

THEOREM XXXVII. *Let $\phi(z)$ be analytic and of exponential type in the sector $|\text{am } z| \leqq \frac{1}{2}\pi$. Let*[4]

(24.01) $$\phi(iy) = O(1), \qquad |y| \to \infty.$$

Let $\{z_n\}$ be a sequence of complex numbers such that

(24.02) $$\lim_{n\to\infty} \frac{n}{z_n} = D \geqq 0$$

and such that for some $d > 0$

(24.03) $$|z_n - z_m| \geqq |n - m| d.$$

A necessary and sufficient condition that

(24.04) $$\limsup_{n\to\infty} \frac{\log|\phi(z_n)|}{|z_n|} = \limsup_{x\to\infty} \frac{\log|\phi(x)|}{x},$$

is that

(24.05) $$\sum_1^\infty \frac{1}{|z_n|} = \infty.$$

[4] The condition (24.01) can easily be replaced by

$$\int_{-\infty}^{\infty} \frac{\log^+|\phi(iy)|}{1+y^2}\,dy < \infty,$$

as will be pointed out in the proof of this theorem.

If $D > 0$ in (24.02), this theorem is an immediate consequence of Theorem XXXIII. However if $D = 0$ Theorem XXXIII no longer gives any results. Thus if $z_n = n \log (n + 1)$, (24.05) is satisfied and Theorem XXXVII applies whereas the theorems of §23 give no results.

Another theorem proved here is

THEOREM XXXVIII. *Let $\phi(z)$ be analytic and of exponential type in the half-plane $|\operatorname{am} z| \leqq \frac{1}{2}\pi$. Let*

$$\phi(iy) = O(e^{\pi L|y|}). \tag{24.06}$$

Let $\{\lambda_n\}$ be an increasing positive sequence such that

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D, \qquad \lambda_{n+1} - \lambda_n \geqq d > 0. \tag{24.07}$$

Let $\Lambda(u)$ be the number of $\lambda_n < u$. If

$$\int_1^\infty \frac{\Lambda(y) - Ly}{y^2}\, dy = \infty \tag{24.08}$$

and if

$$\Lambda(y) + t(y) > Ly \tag{24.09}$$

for some positive function $t(y)$ satisfying

$$\int_1^\infty \frac{t(y)}{y^2}\, dy < \infty, \tag{24.10}$$

then

$$\limsup_{n\to\infty} \frac{\log |\phi(\lambda_n)|}{\lambda_n} = \limsup_{n\to\infty} \frac{\log |\phi(x)|}{x}. \tag{24.11}$$

Thus if

$$\lim_{y\to\infty} \frac{\Lambda(y)}{y} = D$$

it is possible that $D = L$ and yet that (24.08) and therefore Theorem XXXVIII holds. In §23 it was necessary that $D > L$.

The method of this section can best be presented by giving an alternative proof of Theorem XXXIII of the preceding section.

Alternative proof of Theorem XXXIII. Clearly to prove this theorem it suffices to show that

$$\limsup_{n\to\infty} \frac{\log |\phi(z_n)|}{|z_n|} \leqq 0 \tag{24.12}$$

implies

$$\limsup_{x\to\infty} \frac{\log |\phi(x)|}{x} \leqq 0.$$

We again introduce

$$g(z) = \sum_{1}^{\infty} \frac{\phi(z_n)e^{-\epsilon z_n}}{F'(z_n)(z - z_n)} e^{\epsilon z} F(z) \tag{24.13}$$

which by (24.12) exists for any $\epsilon > 0$. As in §23, $g(z_n) = \phi(z_n)$ and

$$\psi(z) = \frac{\phi(z) - g(z)}{F(z)} \tag{24.14}$$

is analytic and of exponential type for $|\operatorname{am} z| \leqq \frac{1}{2}\pi$. From (24.14)

$$|\psi(iy)| \leqq \left|\frac{\phi(iy)}{F(iy)}\right| + \sum_{1}^{\infty} \left|\frac{\phi(z_n)e^{-\epsilon z_n}}{F'(z_n)(iy - z_n)}\right|. \tag{24.15}$$

Since $D > L$, it follows from (22.05) and (23.06) that

$$\frac{\phi(iy)}{F(iy)} = O(e^{-(1/2)\pi(D-L)|y|}), \qquad |y| \to \infty. \tag{24.16}$$

From the last two results

$$\psi(iy) = O(1/|y|). \tag{24.17}$$

But $\psi(z)$ is of exponential type in the right half-plane. This and (24.17) by a theorem of Phragmén-Lindelof, Theorem C', give

$$\psi(re^{i\theta}) = O\left(\frac{1}{r} e^{Br\cos\theta}\right), \qquad |\theta| \leqq \frac{1}{2}\pi,$$

for some B. From this

$$\frac{\psi(z)e^{-Bz}}{1 + z} = O(1/|z|^2), \qquad |\operatorname{am} z| \leqq \frac{1}{2}\pi. \tag{24.18}$$

Using (24.18), it follows by the Cauchy integral theorem that for $x > 0$

$$\begin{aligned} \frac{\psi(z)e^{-Bz}}{1 + z} &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\psi(s)e^{-Bs}}{1 + s} \frac{ds}{z - s} \\ &= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\psi(s)e^{-Bs}}{1 + s} ds \int_0^{\infty} e^{-u(z-s)} du \\ &= \frac{1}{2\pi i} \int_0^{\infty} e^{-uz} du \int_{-i\infty}^{i\infty} \frac{\psi(s)e^{-Bs}}{1 + s} e^{us} ds. \end{aligned}$$

Or if

$$H(u) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\psi(s)e^{-Bs}}{1 + s} e^{us} du, \tag{24.19}$$

then for $x > 0$

$$\frac{\psi(z)e^{-Bz}}{1 + z} = \int_0^{\infty} H(u)e^{-uz} du. \tag{24.20}$$

Using (24.18) and closing the path of integration to the right in (24.19), it is clear that

$$H(u) = 0, \qquad u < 0. \tag{24.21}$$

On the other hand, if we use (24.14) in (24.19) it becomes

$$H(u) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\phi(s)e^{-Bs}}{F(s)(1+s)} e^{us}\,ds - \sum_1^\infty \frac{\phi(z_n)e^{-\epsilon z_n}}{F'(z_n)} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{(u-B+\epsilon)s}}{(1+s)(s-z_n)}\,ds.$$

Or if $u < B - \epsilon$

$$H(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi(it)e^{-Bit}}{F(it)(1+it)} e^{iut}\,dt - \sum_1^\infty \frac{\phi(z_n)e^{(u-B)z_n}}{F'(z_n)(1+z_n)}.$$

That is, if

$$H_1(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\phi(it)e^{-Bit}}{F(it)(1+it)} e^{iut}\,dt \tag{24.22}$$

and

$$H_2(u) = \sum_1^\infty \frac{\phi(z_n)e^{(u-B)z_n}}{F'(z_n)(1+z_n)}, \tag{24.23}$$

then

$$H(u) = H_1(u) - H_2(u), \qquad u < B - \epsilon. \tag{24.24}$$

The series for $H_2(u)$ converges for $u < B$ and represents an analytic function. This follows from (24.12) and (22.06). *Thus $H_2(u)$ is analytic for $u < B$.*

If $\delta = \frac{1}{2}\pi(D - L)$, then from (24.16)

$$\frac{\phi(iy)}{F(iy)} = O(e^{-\delta|y|}), \qquad |y| \to \infty. \tag{24.25}$$

If $w = u + iv$ then it follows from (24.22) and (24.25) that $H_1(w)$ is defined for $|v| < \delta$. The derivative $H_1'(w)$ also exists in this strip. *Thus $H_1(u)$ is analytic for $(-\infty < u < \infty)$.*

Since $H(u) = H_1(u) - H_2(u)$, it now follows that *$H(u)$ is analytic for $u < B - \epsilon$.* But by (24.21)

$$H(u) = 0, \qquad u < 0.$$

Since $H(u)$ is analytic, $u < B - \epsilon$, it follows that

$$H(u) = 0, \qquad u < B - \epsilon. \tag{24.26}$$

Thus (24.20) becomes

$$\frac{\psi(x)e^{-Bx}}{1 + x} = \int_{B-\epsilon}^{\infty} H(u)e^{-ux}\,du.$$

By (24.18) and (24.19) $H(u)$ is bounded. Thus

$$\frac{\psi(x)e^{-Bx}}{1+x} = O(e^{-x(B-\epsilon)}), \qquad x \to \infty,$$

or

$$\psi(x) = O(e^{2\epsilon x}), \qquad x \to \infty.$$

Since $\phi(x) = g(x) + F(x)\psi(x)$, it follows that

$$\limsup_{x\to\infty} \frac{\log|\phi(x)|}{x} \leq 2\epsilon.$$

Since ϵ can be taken arbitrarily small, we have completed the alternative proof of Theorem XXXIII.

The essential difference of this proof from those of §23 is that here we get $\psi(z)$ in terms of $H(u)$. In this section we are attempting to refine the theorems of §23 so that $D > L$ is no longer necessary. We shall now see how dropping $D > L$ affects the argument of the alternative proof just given. A perusal will show that the full force of $D > L$ is used only in the paragraph containing (24.25) where we prove that $H_1(u)$ is analytic, $(-\infty < u < \infty)$. Clearly if $D = L$, (24.25) will no longer hold and $H_1(u)$ need no longer be analytic. That $H_2(u)$ is analytic for $u < B$ regardless of how D compares with L follows from (24.23). Thus the question becomes this. Is there any weaker condition than analyticity that can be imposed on $H_1(u)$ that together with

$$H_1(u) = H_2(u), \qquad u < 0,$$

and $H_2(u)$ analytic for $u < B$, implies

$$H_1(u) = H_2(u), \qquad u < B?$$

That there are such conditions was proved in Chapter V. In particular Theorem XXIV will be of interest here. This theorem states that *if $G(t) \in L(-\infty, \infty)$ and if for $t \to \infty$*

$$G(t) = O(e^{-\theta(t)}), \tag{24.27}$$

where $\theta(t)$ is monotone increasing and

$$\int_1^\infty \frac{\theta(t)}{t^2}\,dt = \infty, \tag{24.28}$$

then if the Fourier transform of $G(t)$ coincides with an analytic function over some interval it coincides with the analytic function over its entire interval of analyticity.

In order to apply this theorem to $H_1(u)$, it follows from (24.22) that it must be shown that $\phi(it)/F(it)$ satisfies (21.27). This is the object of the following lemma.

LEMMA 24.1. *If*

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

and if $\{\lambda_n\}$ *satisfies the requirements of Theorem* XXXVIII, *then there exists an even function* $\theta(u)$, *increasing for* $u > 0$, *such that*

$$\int_1^\infty \frac{\theta(y)}{y^2}\, dy = \infty \tag{24.29}$$

and an even function $t_1(y)$, *increasing for* $y > 0$, *such that*

$$\int_1^\infty \frac{t_1(y)}{y^2}\, dy < \infty, \tag{24.30}$$

and

$$\frac{e^{\pi L|y|}}{F(iy)} = O(e^{\theta(y)+t_1(y)}), \qquad |y| \to \infty. \tag{24.31}$$

Proof of Lemma 24.1. Clearly

$$\log|F(iy)| = \sum_1^\infty \log\left(1 + \frac{y^2}{\lambda_n^2}\right) = \int_0^\infty \log\left(1 + \frac{y^2}{u^2}\right) d\Lambda(u)$$
$$= 2\int_0^\infty \frac{\Lambda(u)}{u} \frac{y^2}{u^2 + y^2}\, du.$$

Since

$$2\int_0^\infty \frac{y^2}{u^2 + y^2}\, du = \pi\,|y|,$$

it follows that

$$\log|F(iy)| - \pi L|y| = 2\int_0^\infty \frac{\Lambda(u) - Lu}{u} \frac{y^2}{u^2 + y^2}\, du.$$

Or

$$\begin{aligned}\log|F(iy)| - \pi L|y| &\geqq 2\int_1^\infty \frac{\Lambda(u) - Lu + t(u)}{u} \frac{y^2}{u^2 + y^2}\, du \\ &\qquad - 2\int_1^\infty \frac{t(u)}{u} \frac{y^2}{u^2 + y^2}\, du - 2L.\end{aligned} \tag{24.32}$$

Let

$$t_1(y) = 2\int_1^\infty \frac{t(u)}{u} \frac{y^2}{u^2 + y^2}\, du. \tag{24.33}$$

Recalling $\Lambda(y) + t(y) > Ly$, (24.32) becomes

$$\log|F(iy)| - \pi L|y| \geqq 2\int_1^y \frac{\Lambda(u) - Lu + t(u)}{u} \frac{y^2}{u^2 + y^2}\, du - t_1(y) - 2L$$
$$\geqq \int_1^y \frac{\Lambda(u) - Lu + t(u)}{u}\, du - t_1(y) - 2L.$$

If we set

$$(24.34) \qquad \theta(y) = \int_1^y \frac{\Lambda(u) - Lu + t(u)}{u}\,du, \qquad y > 1,$$

$\theta(-y) = \theta(y)$, then

$$\log |F(iy)| - \pi L |y| \geqq \theta(y) - t_1(y) - 2L,$$

which gives (24.31).

From (24.34), $\theta(y)$ is increasing and

$$\int_1^\infty \frac{\theta(y)}{y^2}\,dy = \int_1^\infty \frac{dy}{y^2}\int_1^y \frac{\Lambda(u) - Lu + t(u)}{u}\,du$$

$$\geqq \int_1^\infty \frac{dy}{y^2}\int_1^y \frac{\Lambda(u) - Lu}{u}\,du = \int_1^\infty \frac{\Lambda(u) - Lu}{u}\,du \int_u^\infty \frac{dy}{y^2}$$

$$= \int_1^\infty \frac{\Lambda(u) - Lu}{u^2}\,du = \infty.$$

This proves (24.29).

From (24.33)

$$t_1'(y) = 2\int_1^\infty \frac{t(u)}{u}\,\frac{2yu^2}{(u^2+y^2)^2}\,du > 0.$$

Thus $t_1(y)$ is increasing for $y > 0$. Also

$$\int_0^\infty \frac{t_1(y)}{y^2}\,dy = 2\int_0^\infty \frac{dy}{y^2}\int_1^\infty \frac{t(u)}{u}\,\frac{y^2}{u^2+y^2}\,du$$

$$= 2\int_1^\infty \frac{t(u)}{u}\,du\int_0^\infty \frac{dy}{u^2+y^2} = \pi\int_1^\infty \frac{t(u)}{u^2}\,du < \infty,$$

which completes the proof of the lemma

Proof of Theorem XXXVIII. If $t_1(u)$ satisfies the requirements of Lemma 24.1, then by Theorem XXVI for any $\delta > 0$ there exists a function $K(w)$ such that

$$(24.35) \qquad K(u) = O\left(\frac{e^{-t_1(2u)}}{1+u^4}\right),$$

and the Fourier transform of $K(u)$, $k(x)$ vanishes outside of $(-\delta, \delta)$. Thus

$$(24.36) \qquad k(x) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} K(u)e^{-iux}\,du.$$

Let

$$(24.37) \qquad K_1(z) = \frac{1}{(2\pi)^{1/2}}\int_{-\delta}^{\delta} k(t)\bar{k}(t)e^{zt}\,dt.$$

Then by (24.36)

$$K_1(iy) = \frac{1}{2\pi}\int_{-\delta}^{\delta} \bar{k}(t)e^{iyt}\,dt \int_{-\infty}^{\infty} K(u)e^{-iut}\,du$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} K(u)\,du \int_{-\delta}^{\delta} \bar{k}(t)e^{i(y-u)t}\,dt$$
$$= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} K(u)\bar{K}(u-y)\,du.$$

Or for $y > 0$,

$$| K_1(iy) | \leqq \frac{1}{(2\pi)^{1/2}} \max_{-\infty<u<y/2} | \bar{K}(u-y) | \int_{-\infty}^{y/2} | K(u) |\,du$$
$$+ \max_{y/2<u<\infty} | K(u) | \int_{y/2}^{\infty} | \bar{K}(u-y) |\,du$$
$$\leqq \frac{2}{(2\pi)^{1/2}} \max_{|u|\geqq y/2} | K(u) | \int_{-\infty}^{\infty} | K(u) |\,du.$$

Using (24.35) this gives

(24.38) $$K_1(iy) = O\left(\frac{e^{-t_1(y)}}{y^4}\right), \qquad | y | \to \infty.$$

On the other hand by (24.37)

$$K_1(x) \geqq \frac{e^{-\delta x}}{(2\pi)^{1/2}}\int_{-\delta}^{\delta} | k(t) |^2\,dt, \qquad x > 0,$$

or

(24.39) $$\frac{1}{K_1(x)} = O(e^{\delta x}), \qquad x \to \infty.$$

Also from (24.37)

(24.40) $$K_1(z) = O(e^{\delta|x|}).$$

As in the alternative proof of Theorem XXXIII it suffices to show that

(24.41) $$\lim_{n\to\infty} \frac{\log | \phi(\lambda_n) |}{\lambda_n} \leqq 0$$

implies

(24.42) $$\limsup_{x\to\infty} \frac{\log | \phi(x) |}{x} \leqq 0.$$

We define

(24.43) $$g(z) = \sum_{1}^{\infty} \frac{\phi(\lambda_n)e^{-\epsilon\lambda_n}}{F'(\lambda_n)(z-\lambda_n)}\, e^{\epsilon z}F(z)$$

with $\epsilon > \delta$, and

$$\psi(z) = \frac{\phi(z) - g(z)}{F(z)}. \tag{24.44}$$

Then as before $g(z)$ is an entire function and $\psi(z)$ is analytic and of exponential type for $|\operatorname{am} z| \leqq \frac{1}{2}\pi$. From (24.44)

$$|\psi(iy)| \leqq \left|\frac{\phi(iy)}{F(iy)}\right| + \sum_1^\infty \left|\frac{\phi(z_n)e^{-\epsilon z_n}}{F'(z_n)(iy - z_n)}\right|. \tag{24.45}$$

By (24.40) $K_1(z)\psi(z)$ is also of exponential type for $|\operatorname{am} z| \leqq \frac{1}{2}\pi$. By (24.38) and (24.45)

$$K_1(iy)\psi(iy) = O\left(\frac{e^{-\epsilon_1(y)}|\phi(iy)|}{|F(iy)|\, y^4} + \frac{1}{|y|^5}\right), \qquad |y| \to \infty.$$

But $\phi(iy) = O(e^{\pi l|y|})$. Thus using (24.31),

$$K_1(iy)\psi(iy) = O\left(\frac{e^{-\theta(y)}}{y^4} + \frac{1}{|y|^5}\right) = O\left(\frac{1}{y^4}\right). \tag{24.46}$$

It now follows from Theorem C′ of Phragmén-Lindelof that for some B

$$K_1(z)\psi(z) = O\left(\frac{e^{Bx}}{|z|^4}\right), \qquad |\operatorname{am} z| \leqq \tfrac{1}{2}\pi,$$

or

$$K_1(z)\psi(z)e^{-Bz} = O\left(\frac{1}{|z|^4}\right), \qquad |\operatorname{am} z| \leqq \tfrac{1}{2}\pi. \tag{24.47}$$

Using (24 47) it follows by Cauchy's integral theorem that for $x > 0$,

$$\begin{aligned} K_1(z)\psi(z)e^{-Bz} &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{K_1(s)\psi(s)e^{-Bs}}{z - s}\, ds \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K_1(s)\psi(s)e^{-Bs}\, ds \int_0^\infty e^{-u(z-s)}\, du \\ &= \frac{1}{2\pi i}\int_0^\infty e^{-uz}\, ds \int_{-i\infty}^{i\infty} K_1(s)\psi(s)e^{-Bs+us}\, ds. \end{aligned}$$

Here we are following almost identically the alternative proof of Theorem XXXIII. We define

$$H(u) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K_1(s)\psi(s)e^{-Bs+us}\, ds. \tag{24.48}$$

Then

$$K_1(z)\psi(z)e^{-Bz} = \int_0^\infty H(u)e^{-uz}\, du.$$

Using (24.39) this gives

$$\psi(x) = O\left(e^{(B+\delta)x}\int_0^\infty |H(u)|\, e^{-ux}\, du\right), \qquad x \to \infty. \tag{24.49}$$

Using (24.47) in (24.48) we can close the path of integration to the right and obtain

$$H(u) = 0, \qquad u < 0,$$

On the other hand, using the definition of $\psi(z)$ in (24.48) gives

$$H(u) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{K_1(s)\phi(s)e^{-Bs}}{F(s)} e^{us}\, ds - \sum_1^\infty \frac{\phi(\lambda_n)e^{-\epsilon\lambda_n}}{F'(\lambda_n)} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{K_1(s)e^{(u-B+\epsilon)s}}{s - \lambda_n}\, ds.$$

By (24.40) this becomes

$$H(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K_1(it)\phi(it)e^{-iBt}}{F(it)} e^{iut}\, dt - \sum_1^\infty \frac{K_1(\lambda_n)\phi(\lambda_n)e^{(u-B)\lambda_n}}{F'(\lambda_n)}$$

for $u < B - \epsilon - \delta$. Or

$$H(u) = H_1(u) - H_2(u), \qquad u < B - 2\epsilon' \tag{24.50}$$

where

$$H_1(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{K_1(it)\phi(it)e^{-iBt}}{F(it)} e^{iut}\, dt, \tag{24.51}$$

and

$$H_2(u) = \sum_1^\infty \frac{K_1(\lambda_n)\phi(\lambda_n)e^{(u-B)\lambda_n}}{F'(\lambda_n)}, \qquad u < B - 2\epsilon. \tag{24.52}$$

The series for $H_2(u)$ converges for $u < B - \delta$ and therefore certainly for $u < B - 2\epsilon$ and represents an analytic function. Thus $H_2(u)$ is analytic for $u < B - 2\epsilon$.

Since $\phi(it) = O(e^{\pi L|t|})$, using (24.31) and (24.38) gives

$$\frac{K_1(it)\phi(it)}{F(it)} = O\left(\frac{e^{-\theta(t)}}{t^4}\right), \qquad |t| \to \infty.$$

Thus if

$$G(t) = (2\pi)^{1/2}\frac{K_1(it)\phi(it)e^{-iBt}}{F(it)}$$

then $G(t)$ satisfies (24.27) and (24.28); and by (24.51), $H_1(u)$ is the Fourier transform of $G(t)$. Since we have shown $H(u) = 0$, $u < 0$, it follows that

$$H_1(u) = H_2(u), \qquad u < 0.$$

But $H_2(u)$ is analytic, $u < B - 2\epsilon$, and $H_1(u)$ is the transform of $G(t)$. Thus from the remarks following (24.28)

$$H_1(u) = H_2(u), \qquad u < B - 2\epsilon,$$

or $H(u) = 0$, $(u < B - 2\epsilon)$. Using this in (24.49) gives

$$\psi(x) = O\left(e^{(B+\delta)x}\int_{B-2\epsilon}^{\infty} |H(u)|\, e^{-ux}\,du\right), \qquad x \to \infty.$$

By (24.46) and (24.48), $H(u)$ is bounded. Thus

$$\psi(x) = O(e^{(\delta+2\epsilon)x}) = O(e^{3\epsilon x}), \qquad x \to \infty,$$

since $\epsilon > \delta$. But $\phi(x) = g(x) + F(x)\phi(x)$ and therefore $\phi(x) = O(e^{4\epsilon x})$. δ can be chosen arbitrarily small and therefore so can ϵ. Therefore (24.42) follows and the theorem is proved.

In much the same way the following theorem is proved.

Theorem XXXVIII-A. *Theorem* XXXVIII *remains true if* (24.06) *and* (24.08) *are both replaced by*

$$\phi(iy) = O(e^{\pi L|y| - \theta(y)}), \qquad |y| \to \infty,$$

for an even function $\theta(y)$ increasing for $y > 0$ and satisfying

$$\int_1^{\infty} \frac{\theta(y)}{y^2}\,dy = \infty.$$

We now turn to the proof of Theorem XXXVII.

Lemma 24.2. *Let*

(24.53) $$F(z) = \prod_1^{\infty}\left(1 - \frac{z^2}{z_n^2}\right)$$

where $\{z_n\}$ satisfies the requirements of Theorem XXXVII. *Then*

(24.54) $$\frac{1}{F(iy)} = O(e^{-\theta(y)}), \qquad |y| \to \infty,$$

where $\theta(y)$ is even and increasing for $y > 0$, and

(24.55) $$\int_1^{\infty} \frac{\theta(y)}{y^2}\,dy = \infty.$$

Proof of Lemma 24.2. If $z_n = r_n e^{i\theta_n}$ there is no loss of generality in assuming that $\theta_n \leq \frac{1}{4}\pi$. We have, if $\zeta(u)$ is the number of $|z_n| < u$,

$$\begin{aligned}\log|F(iy)| &= \sum_1^{\infty} \log\left(1 + \frac{2y^2}{r_n^2}\cos 2\theta_n + \frac{y^4}{r_n^4}\right)^{1/2} \\ &\geq \tfrac{1}{2}\sum_1^{\infty}\log\left(1 + \frac{y^4}{r_n^4}\right) = \frac{1}{2}\int_0^{\infty}\log\left(1 + \frac{y^4}{u^4}\right)d\zeta(u) \\ &> \frac{\log 2}{2}\int_0^{|y|} d\zeta(u) > \tfrac{1}{4}\zeta(|y|).\end{aligned}$$

Thus if we take $\theta(y) = \frac{1}{4}\zeta(|y|)$, then (24.54) is satisfied. Also

$$\int_1^A \frac{\theta(y)}{y^2}\,dy = \frac{1}{4}\int_1^A \frac{\zeta(y)}{y^2}\,dy \geqq \int_1^A \frac{d\zeta(y)}{y} - \frac{1}{4}\frac{\zeta(A)}{A}.$$

Since $n/z_n \to D$, letting $A \to \infty$ we have

$$\int_1^\infty \frac{\theta(y)}{y^2}\,dy \geqq \frac{1}{4}\int_1^\infty \frac{d\zeta(y)}{y} - \tfrac{1}{4}D = \tfrac{1}{4}\sum_1^\infty \frac{1}{|z_n|} - \tfrac{1}{4}D = \infty.$$

This proves (24.55) and completes the proof of the lemma.

Proof of Theorem XXXVII. The proof follows word for word the alternative proof of Theorem XXXIII except for the facts that (24.16) and (24.25) are both replaced by

$$\frac{\phi(iy)}{F(iy)} = O(e^{-\theta(y)}), \qquad |y| \to \infty, \tag{24.56}$$

and that $H_1(u)$ is no longer analytic, which nullifies the argument from (24.25) to (24.26). Equation (24.56) follows from $\phi(iy) = O(1)$ and (24.54).

To replace the argument from (24.25) to (24.26), we recall that

$$H_1(u) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\phi(it)e^{-Bit}}{F(it)(1+it)}\,e^{iut}\,dt.$$

If we set

$$G(t) = \frac{1}{(2\pi)^{1/2}}\frac{\phi(it)e^{-Bit}}{F(it)(1+it)},$$

then $H_1(u)$ is the Fourier transform of $G(t)$ and by (24.56), $G(t)$ satisfies (24.27). Thus by the argument following (24.27), since by (24.21) $H(u) = 0$, $(u < 0)$, or in other words $H_1(u) = H_2(u)$, $(u < 0)$, and since $H_2(u)$ is analytic for $u < B - \epsilon$, it follows that

$$H_1(u) = H_2(u), \qquad u < B - \epsilon,$$

or in other words that $H(u) = 0$, $(u < B - \epsilon)$. This is (24.26), and the argument now again follows exactly the alternative proof of Theorem XXXIII.[5]

[5] In case $\phi(iy)$ is not bounded but satisfies

$$\int_{-\infty}^\infty \frac{\log^+|\phi(iy)|}{1+y^2}\,dy < \infty,$$

a function $K_1(z)$, analytic and bounded in the right half-plane, is introduced which satisfies $\log|K_1(iy)| = -\log^+|\phi(iy)|$. $K_1(z)$ is used in much the same way as a similar function also designated by $K_1(z)$ is used in the proof of Theorem XXXIII. $K_1(z) = e^{-u(x,y)-iv(x,y)}$ where

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{x\log^+|\phi(i\eta)|}{x^2+(y-\eta)^2}\,d\eta$$

and $v(x, y)$ is the conjugate function to $u(x, y)$.

We now turn to the other part of the proof and show that

$$\sum_1^{\infty} \frac{1}{|z_n|} = \infty \tag{24.57}$$

is a necessary condition. Let us assume

$$\sum_1^{\infty} \frac{1}{|z_n|} < \infty. \tag{24.58}$$

Then

$$\phi(z) = \prod_1^{\infty} \frac{(z - z_n)(z - \bar{z}_n)}{(z + z_n)(z + \bar{z}_n)} = \prod_1^{\infty} \left(1 - \frac{2z}{z_n + z}\right)\left(1 - \frac{2z}{\bar{z}_n + z}\right)$$

exists and is analytic in the right half-plane, since, for $x > 0$, (24.58) gives

$$\sum_1^{\infty} \left|\frac{2z}{z_n + z}\right| < \infty, \qquad \sum_1^{\infty} \left|\frac{2z}{\bar{z}_n + z}\right| < \infty.$$

Since for $x > 0$

$$\left|\frac{z - z_n}{z + z_n}\right| \leqq 1, \qquad \left|\frac{z - \bar{z}_n}{z + \bar{z}_n}\right| \leqq 1,$$

it follows that

$$|\phi(z)| \leqq 1, \qquad |\theta| \leqq \tfrac{1}{2}\pi. \tag{24.59}$$

Thus $\phi(z)$ satisfies the requirements of Theorem XXXVII. But $\phi(z_n) = 0$, and thus,

$$\lim_{n\to\infty} \frac{\log|\phi(z_n)|}{|z_n|} = -\infty. \tag{24.60}$$

If (24.58) were sufficient for Theorem XXXVII, (24.60) would imply

$$\lim_{x\to\infty} \frac{\log|\phi(x)|}{x} = -\infty.$$

But this and (24.59), by Theorem C' of Phragmén-Lindelöf, imply that $\phi(z) \equiv 0$, which is impossible. Thus (24.58) is insufficient for Theorem XXXVII, or in other words (24.57) is necessary. This completes the proof of Theorem XXXVII.

Theorem XXXIV can be refined by the use of Theorem XXXVIII to give the following:

THEOREM XXXIX. *Let $\Phi(z)$ be an analytic function in the right half-plane $|\operatorname{am} z| \leqq \frac{1}{2}\pi$ such that for any $\epsilon > 0$*

$$\Phi(re^{i\theta}) = O\left(\exp\{(a \log r \cos\theta + \pi b|\sin\theta| + \epsilon\cos\theta)r\}\right), \quad |\theta| \leqq \tfrac{1}{2}\pi, \tag{24.61}$$

where $a \geqq 0$, $b \geqq -\frac{1}{2}a$. Let $\{\lambda_n\}$ be a positive sequence satisfying

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = D, \qquad \lambda_{n+1} - \lambda_n \geqq d > 0.$$

Let $\Lambda(u)$ *be the number of* $\lambda_n < u$. *If*

$$\int_1^\infty \frac{\Lambda(u) - (b + \frac{1}{2}a)u}{u^2}\, du = \infty$$

and if for some positive $t(u)$ *such that*

$$\int_1^\infty \frac{t(u)}{u^2}\, du < \infty$$

the inequality

$$\Lambda(u) + t(u) \geqq (b + \tfrac{1}{2}a)u$$

holds, then

(24.62) $$\limsup_{n\to\infty} \frac{\log |\Phi(\lambda_n)|}{\lambda_n} \leqq \pi p$$

implies

(24.63) $$\Phi(re^{i\theta}) = O(\exp \{\pi r(p \cos \theta + b \,|\sin \theta| + \epsilon)\}), \qquad |\theta| \leqq \tfrac{1}{2}\pi,$$

for any $\epsilon > 0$.

Proof of Theorem XXXIX. As in the proof of Theorem XXXIV, we consider

$$\phi(z) = \frac{\Phi(z)}{\Gamma(1 + az)}.$$

Proceeding as in Theorem XXXIV, it follows from Theorem XXXVIII that

$$\lim_{x\to\infty} \frac{\log |\phi(x)|}{x} = -\infty.$$

Then $g(z)$ is defined as in (23.25) and

$$\psi(z) = \frac{\phi(z) - g(z)}{F(z)}.$$

However, instead of working with $\psi(z)$ we work with $\psi(z)K_1(z)$ where $K_1(z)$ is defined as in the proof of Theorem XXXVIII. Thus (23.27) is replaced by

$$|\psi(iy)K_1(iy)| \leqq M_1 + \sum_1^\infty \left| \frac{\phi(\lambda_n)K_1(\lambda_n)e^{A\lambda_n}}{F'(\lambda_n)} \right|.$$

Proceeding as in Theorem XXXIV with $\psi(z)K_1(z)$ instead of $\psi(z)$, there are no further difficulties.

Related to Theorem XXXIX in exactly the same way as Theorem XXXV is related to Theorem XXXIV is the following theorem.

THEOREM XL. *If, in Theorem* XXXIX, (24.62) *is replaced by*

$$\Phi(\lambda_n) = O(e^{-k\lambda_n \log \lambda_n}), \qquad n \to \infty,$$

with $k > 0$, *then* (24.63) *is replaced by*

$$\Phi(re^{i\theta}) = O(\exp\{(-k \log r \cos\theta + \epsilon \cos\theta + \pi b \mid \sin\theta \mid)r\}), \qquad \mid \theta \mid \leqq \tfrac{1}{2}\pi.$$

The proof of this theorem follows closely that of Theorem XXXIX in just the same way as the proof of Theorem XXXV follows closely that of Theorem XXXIV. A sharper result than Theorem XXXVI is the following theorem.

THEOREM XLI. *If, in Theorem* XXXIX, (24.62) *is replaced by*

$$\lim_{n\to\infty} \frac{\log \mid \Phi(\lambda_n) \mid + 2b\lambda_n \log \lambda_n}{\lambda_n} = -\infty, \tag{24.64}$$

then $\Phi(z) \equiv 0$.

Proof of Theorem XLI. Applying (24.64) to $e^{Bz}\Phi(z)$, $(B > 0)$, it follows that

$$e^{B\lambda_n}\Phi(\lambda_n) = O(e^{-2b\lambda_n \log \lambda_n}).$$

Applying Theorem XL to $e^{Bz}\Phi(z)$, it follows that

$$\mid \Phi(z) \mid \leqq B_1 \exp\{(-2b \log r \cos\theta - B\cos\theta + \pi b \mid \sin\theta \mid)r\}, \quad \mid \theta \mid \leqq \tfrac{1}{2}\pi, \tag{24.65}$$

where B_1 is a constant depending on B. Applying Carleman's theorem, Theorem B, to $\Phi(z)$, we have, if $\Phi(z) \not\equiv 0$,

$$-A_1 \leqq \frac{1}{2\pi}\int_1^R \log \mid \Phi(iy)\Phi(-iy) \mid \left(\frac{1}{y^2} - \frac{1}{R^2}\right) dy + \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \log \mid \Phi(Re^{i\theta}) \mid \cos\theta \, d\theta,$$

where A_1 depends only on $\Phi(z)$. Using (24.61), this becomes

$$-A_2 \leqq b\int_1^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right) y \, dy + \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \log \mid \Phi(re^{i\theta}) \mid \cos\theta \, d\theta$$

where A_2 depends only on $\Phi(z)$ Applying (24.65) this becomes

$$-A_2 \leqq b \log R + \frac{-2b \log R - B + G}{\pi}\int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta$$
$$+ b\int_{-\pi/2}^{\pi/2} \mid \sin\theta \mid \cos\theta \, d\theta + \frac{\log B_1}{\pi R}\int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta.$$

Or

$$-A_2 \leqq -\tfrac{1}{2}B + \tfrac{1}{2}\epsilon + b + \frac{2}{\pi R}\log B_1.$$

Letting $R \to \infty$ we see that by choosing

$$B > 2A_2 + 2b + \epsilon$$

we obtain a contradiction. Thus $\Phi(z) \equiv 0$.

25.[6] An extension of a theorem of Iyer. Here we use the method of §23 on another problem.

THEOREM XLII.[7] *Let* $\{z_n\}$ *and* $\{w_n\}$ *be two sequences of complex numbers such that*

$$\lim_{n\to\infty} \frac{n}{z_n} = D_1 > 0, \qquad \lim_{n\to\infty} \frac{n}{w_n} = D_2 > 0,$$

and for some $d > 0$,

$$| z_n - z_m | \geqq | n - m | d, \qquad | w_n - w_m | \geqq | n - m | d.$$

Let $f(z)$ *be an entire function such that*

$$f(\pm z_n) = O(1), \qquad f(\pm iw_n) = O(1), \qquad n \to \infty, \tag{25.01}$$

and

$$\limsup_{|z|\to\infty} \frac{\log |f(z)|}{|z|} = k < \pi(D_1^2 + D_2^2)^{1/2}. \tag{25.02}$$

Then $f(z)$ *is a constant.*

That the theorem is best possible follows from considering $f(z) = \sin \pi D_1 z \sinh \pi D_2 z$ with $z_n = n/D_1$ and $w_n = n/D_2$.

Proof of Theorem XLII. Let

$$a = \pi D_1, \qquad b = \pi D_2, \qquad \alpha = \tan^{-1} a/b, \tag{25.03}$$

and let

$$F_1(z) = \prod_1^\infty \left(1 - \frac{z^2}{z_n^2}\right), \qquad F_2(z) = \prod_1^\infty \left(1 + \frac{z^2}{w_n^2}\right).$$

Then $F_1(z)$ and $F_2(iz)$ satisfy the requirements of Theorem XXXI and thus we have

$$\frac{1}{F_1'(\pm z_n)} = O(e^{\epsilon |z_n|}), \qquad \frac{1}{F_2'(\pm iw_n)} = O(e^{\epsilon |w_n|}), \tag{25.04}$$

[6] Cf. Levinson. *Integral functions bounded on sequences of points*, Duke Mathematical Journal, vol. 4 (1938), p. 170.

[7] Iyer proved this theorem for real sequences $\{z_n\}$ and $\{w_n\}$ with (25.02) replaced by the more restrictive condition

$$k < \pi \min (D_1, D_2)$$

or else with (25.01) replaced by

$$\lim_{n\to\infty} \frac{\log | f(\pm z_n) |}{| z_n |} = -\infty, \qquad \lim_{n\to\infty} \frac{\log | f(\pm iw_n) |}{| w_n |} = -\infty.$$

V. G. Iyer, *On the order and type of integral functions bounded at a sequence of points*, Annals of Mathematics, vol. 38 (1937), p. 311.

$$(25.05)\qquad F_1(re^{i\theta}) = O(\exp\{ar\,|\sin\theta| + \epsilon r\}),$$
$$F_2(re^{i\theta}) = O(\exp\{br\,|\cos\theta| + \epsilon r\}),$$

$$(25.06)\qquad \frac{1}{F_1(re^{i\theta})} = O(\exp\{-ar\,|\sin\theta| + \epsilon r\}), \qquad |z \pm z_n| \geqq \tfrac{1}{8}d.$$

$$(25.07)\qquad \frac{1}{F_2(re^{i\theta})} = O(\exp\{-br\,|\cos\theta| + \epsilon r\}), \qquad |z \pm iw_n| \geqq \tfrac{1}{8}d.$$

For $\gamma > 0$ let

$$(25.08)\qquad \begin{aligned}\psi(z) = \frac{f(z)}{F_1(z)F_2(z)} &- \sum_1^\infty \frac{f(z_n)\exp\{(z - z_n)(-b - ib^2/a + \gamma)\}}{F_1'(z_n)F_2(z_n)(z - z_n)} \\ &- \sum_1^\infty \frac{f(-iw_n)\exp\{(z + iw_n)(-ia - a^2/b + i\gamma)\}}{F_1(-iw_n)F_2'(-iw_n)(z + iw_n)}.\end{aligned}$$

The series converge by (25.01), (25.04), (25.06) and (25.07). For any $\epsilon > 0$ and small $\delta > 0$,

$$(25.09)\qquad \begin{aligned}\psi(re^{i(\alpha-\delta)}) = O(&\exp\{-\{a\sin(\alpha-\delta) + b\cos(\alpha-\delta) - k - \epsilon\}r\} \\ &+ \exp\{-(b-\gamma)r\cos(\alpha-\delta) + (b^2/a)r\sin(a-\delta)\} \\ &+ \exp\{(a-\gamma)r\sin(\alpha-\delta) - (a^2/b)r\cos(\alpha-\delta)\}).\end{aligned}$$

Since $\tan\alpha = a/b$, simple calculations give

$$a\sin(\alpha-\delta) + b\cos(\alpha-\delta) = (a^2+b^2)^{1/2}\cos\delta,$$

$$-b\cos(\alpha-\delta) + \frac{b^2}{a}\sin(\alpha-\delta) = -(a^2+b^2)^{1/2}\frac{b}{a}\sin\delta,$$

$$a\sin(\alpha-\delta) - \frac{a^2}{b}\cos(\alpha-\delta) = -(a^2+b^2)^{1/2}\frac{a}{b}\sin\delta.$$

Thus (25.09) becomes

$$(25.10)\qquad \begin{aligned}\psi(re^{i(\alpha-\delta)}) = O(&\exp\{-[(a^2+b^2)^{1/2}\cos\delta - k - \epsilon]r\} \\ &+ \exp\{\gamma r - (a^2+b^2)^{1/2}(b/a)r\sin\delta\} \\ &+ \exp\{\gamma r - (a^2+b^2)^{1/2}(a/b)r\sin\delta\}).\end{aligned}$$

There is no essential difference between the cases $a > b$, $b > a$. Here we assume that $a \geqq b$. Then if

$$p = (a^2+b^2)^{1/2}b/a,$$

it follows that

$$p \leqq (a^2+b^2)^{1/2}a/b$$

and (25.10) becomes

$$\psi(re^{i(\alpha-\delta)}) = O(\exp\{-(a^2+b^2)^{1/2}r\cos\delta + (k+\epsilon)r\} + \exp\{\gamma r - pr\sin\delta\}). \tag{25.11}$$

Let us choose $\epsilon = \frac{1}{2}[(a^2+b^2)^{1/2} - k]$. Then since $(a^2+b^2)^{1/2} > k$ we can choose $\delta > 0$ so small that

$$(a^2+b^2)^{1/2}\cos\delta - k - \epsilon > p\sin\delta.$$

Thus (25.11) becomes

$$\psi(re^{i(\alpha-\delta)}) = O(e^{\gamma r - pr\sin\delta}).$$

We now introduce a small $\eta > 0$ and take $\gamma = \eta\sin\delta$. Then

$$\psi(re^{i(\alpha-\delta)}) = O(e^{-(p-\eta)r\sin\delta}). \tag{25.12}$$

Similarly

$$\psi(re^{i(-\pi+\alpha+\delta)}) = O(e^{-(p-\eta)r\sin\delta}). \tag{25.13}$$

From (25.08), $\psi(z)$ is analytic and of exponential type for $-\pi + \alpha < \operatorname{am} z < \alpha$. But by (25.12) and (25.13),

$$\psi(z)e^{ize^{-i\alpha}(p-\eta)} \tag{25.14}$$

is bounded along the lines $\operatorname{am} z = \alpha - \delta$ and $\operatorname{am} z = -\pi + \alpha + \delta$. Thus by Theorem C of Phragmén-Lindelof, (25.14) is bounded in the entire sector $-\pi + \alpha + \delta \leqq \operatorname{am} z \leqq \alpha - \delta$. In particular then it is bounded for $\operatorname{am} z = 0$. Thus

$$\psi(x) = O(e^{-(p-\eta)x\sin\alpha}), \qquad x \to \infty.$$

Recalling $p = (a^2+b^2)^{1/2}(b/a)$ and (25.05), this gives

$$\psi(x)F_1(x)F_2(x) = O(\exp\{\{b + \epsilon + \eta - (a^2+b^2)^{1/2}b\sin\alpha/a\}x\}), \qquad x \to \infty.$$

Since $(a^2+b^2)^{1/2}\sin\alpha = a$, this becomes

$$\psi(x)F_1(x)F_2(x) = O(e^{(\epsilon+\eta)x}), \qquad x \to \infty. \tag{25.15}$$

But by (25.08)

$$f(x) = \psi(x)F_1(x)F_2(x) + F_2(x)e^{-(b-\gamma)x-ib^2x/a}\sum_1^\infty \frac{f(z_n)F_1(x)e^{z_n(b-\gamma+ib^2/a)}}{F_1'(z_n)F_2(z_n)(x-z_n)}$$
$$+ F_1(x)F_2(x)e^{-i(a-\gamma)x-a^2x/b}\sum_1^\infty \frac{f(-iw_n)e^{w_n(a-\gamma-ia^2/b)}}{F_1(-iw_n)F_2'(-iw_n)(x+iw_n)}.$$

Using (25.15) and (25.05) and recalling that $a \geqq b$, this gives

$$f(x) = O(e^{(3\epsilon+\eta+\gamma)x}), \qquad x \to \infty.$$

But $\gamma = \eta\sin\delta$ and ϵ and η can be chosen arbitrarily close to zero. Thus re-

defining ϵ we obtain $f(x) = O(e^{\epsilon x})$, $(x \to \infty)$, for any x. But $f(-z)$ possesses the same properties as $f(z)$. Thus the result holds for negative x. That is,

(25.16) $$f(x) = O(e^{\epsilon|x|}), \qquad |x| \to \infty.$$

If

$$\limsup_{|y|\to\infty} \frac{\log|f(iy)|}{|y|} = c,$$

then it follows from (25.16) and Theorem C′ of Phragmén-Lindelöf that

$$f(z) = O(e^{cr|\sin\theta| + \epsilon r}).$$

But by Theorem XXXII since $D_2 > 0$,

$$c = \limsup_{n\to\infty} \frac{\log|f(\pm iw_n)|}{|w_n|}.$$

Thus $c = 0$ and

(25.17) $$f(z) = O(e^{\epsilon|z|}).$$

Let $z_{-n} = -z_n$ and $w_{-n} = -w_n$. For any $N > 0$ let

(25.18) $$H_N(z) = \frac{f^N(z)}{F_1(z)F_2(z)} - \sum_{-\infty}^{\infty} \frac{f^N(z_n)}{F_1'(z_n)F_2(z_n)(z - z_n)} - \sum_{-\infty}^{\infty} \frac{f^N(iw_n)}{F_1(iw_n)F_2'(iw_n)(z - iw_n)},$$

where the terms $n = 0$ are omitted from the sums. By (25.06) and (25.07) the series converge and $H_N(z)$ is of exponential type. Also by (25.06), (25.07), and (25.17)

$$H_N(\pm re^{\pm i\alpha}) = O(e^{\{N\epsilon - (a^2+b^2)^{1/2}\}r} + 1/r).$$

Since ϵ can be taken arbitrarily small, it follows that

(25.19) $$H_N(\pm re^{\pm i\alpha}) = o(1), \qquad r \to \infty.$$

Thus $H_N(z)$ is bounded on the lines am $z = \pm\alpha, \pm(\pi - \alpha)$. Thus by Theorem C of Phragmén-Lindelöf, it is bounded in the entire plane and therefore a constant. But by (25.19) this constant must be zero. Thus (25.18) becomes

(25.20) $$f^N(z) = F_1(z)F_2(z)\left(\sum_{-\infty}^{\infty} \frac{f^N(z_n)}{F_1'(z_n)F_2(z_n)(z - z_n)} + \sum_{-\infty}^{\infty} \frac{f^N(iw_n)}{F_1(iw_n)F_2'(iw_n)(z - iw_n)}\right).$$

By (25.01) there exists an M such that

$$|f(\pm z_n)| \leq M, \qquad |f(\pm iw_n)| \leq M.$$

Thus by (25.20) there exists a $C > 1$ independent of N such that

$$|f^N(z)| \leqq CM^N e^{(a+b)r}$$

or

$$|f(z)| \leqq CMe^{(a+b)r/N}.$$

Since N can be chosen arbitrarily large, it follows that

$$|f(z)| \leqq CM.$$

But this means $f(z)$ must be a constant. This completes the proof.

CHAPTER VIII

AN INEQUALITY AND FUNCTIONS OF ZERO TYPE

26. Generalization of a problem of Pólya. In this chapter we shall prove an inequality and show how it leads to a considerable extension of results of Pólya on functions of zero type. The inequality is

THEOREM XLIII. *Let $M(x)$ be a positive even function monotone decreasing for increasing $|x|$. Let $M(x) \to \infty$ as $x \to 0$. Let $f(z)$ be analytic for $(|x| \leqq a, |y| \leqq b)$ and let*

(26.01) $$|f(x+iy)| \leqq M(x), \qquad |x| \leqq a, |y| \leqq b.$$

If

(26.02) $$\int_0^1 \log\log M(x)\,dx < \infty,$$

then there exists a constant C, depending only on $M(x)$ and δ such that

(26.03) $$|f(x+iy)| \leqq C, \qquad |x| \leqq a, |y| \leqq b(1-\delta).$$

We shall show in Chapter IX that Theorem XLIII is a best possible result.[1]

The problem of Pólya which we shall generalize here is the following: *Let $G(z)$ be an entire function such that*

(26.04) $$\limsup_{|z|\to\infty} \frac{\log|G(z)|}{|z|} \leqq 0.$$

If as $n \to \infty$

(26.05) $$G(\pm n) = O(1),$$

then $G(z)$ is a constant.[2]

Roughly what we shall show is that instead of $G(\pm n) = O(1)$ it suffices for

$$G(\pm\lambda_n) = O(1)$$

where $\{\lambda_n\}$ satisfies the requirement

$$\lambda_n - n = O\left(\frac{n}{(\log n)^{1+\delta}}\right), \qquad n \to \infty, \delta > 0.$$

This restriction on $\{\lambda_n\}$ is only a little more stringent than

[1] The results of this and the next chapter are sketched in Abstract 472, Bulletin of the American Mathematical Society, vol. 44 (1938), p. 789, and in Abstract 141, loc. cit., vol. 45 (1939), p. 236.

[2] This problem was set by Pólya, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 40 (1931), Problem 105. Many solutions have been given.

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = 1.$$

However we shall show that $G(z)$ can satisfy

$$G(\pm\lambda_n) = O(1)$$

where

$$\lambda_n - n = O\left(\frac{n}{\log n}\right), \qquad n \to \infty,$$

and yet not be a constant. Here again

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = 1.$$

Thus for $\{\lambda_n\}$ merely to have a density $D > 0$ is not enough for the validity of our results. This result deviates very much from existing density theorems in that it goes as far as it does and yet need not be true for a set $\{\lambda_n\}$ having a density $D > 0$.

In order to illustrate the method of this chapter, let us use it to prove the original result of Pólya, that is, $G(z)$ is a constant if (26.04) and (26.05) are satisfied.

Proof. For any integer $N > 0$, let

$$H_N(w) = \frac{1}{2i} \int_{-i\infty}^{i\infty} \frac{zG^N(z)}{\sin \pi z} e^{wz}\, dz, \tag{26.06}$$

where $w = u + iv$. Equation (26.06) can be written as

$$H_N(w) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{iyG^N(iy)}{\sin \pi iy} e^{iyw}\, dy. \tag{26.07}$$

Since by (26.04) for any $\epsilon > 0$

$$\frac{yG^N(iy)}{\sin \pi iy} = O(e^{-(\pi-\epsilon)|y|}), \qquad |y| \to \infty,$$

it follows from (26.07) that $H_N(w)$ is analytic in the strip $(-\infty < u < \infty, |v| < \pi)$. Again by (26.04) the path of integration in (26.06) can be closed to the right if $u < 0$ and to the left if $u > 0$ for $|v| < \pi$ giving

$$\begin{aligned} H_N(w) &= \sum_{n=1}^{\infty} (-1)^{n-1} nG^N(-n)e^{-nw}, & u > 0, \\ H_N(w) &= \sum_{n=1}^{\infty} (-1)^{n-1} nG^N(n)e^{nw}, & u < 0. \end{aligned} \tag{26.08}$$

From (26.05) it is clear that with no restriction we can assume

$$|G(\pm n)| \leqq 1.$$

Thus by (26.08)

$$|H_N(w)| \leqq \sum_1^\infty ne^{-n|u|}.$$

Or

(26.09) $$|H_N(w)| \leqq 10/u^2.$$

Setting $M(u) = 10/u^2$, it follows from Theorem XLIII that for some $C > 1$ and *independent* of N,

$$|H_N(u)| \leqq C, \qquad |u| < 1.$$

Combining this with (26.09) for $|u| \geqq 1$,

(26.10) $$|H_N(u)| \leqq \frac{20C}{1+u^2}.$$

Applying the Fourier transform theorem to (26.07),

$$\frac{iyG^N(iy)}{\sin \pi iy} = \frac{1}{\pi}\int_{-\infty}^{\infty} H_N(u)e^{-iuy}\,du.$$

By (26.10) this gives

$$\left|\frac{iyG^N(iy)}{\sin \pi iy}\right| \leqq 20C.$$

Or

$$|G(iy)| \leqq \left|\frac{20C \sin \pi iy}{y}\right|^{1/N}.$$

Letting $N \to \infty$

(26.11) $$|G(iy)| \leqq 1.$$

It follows from this and (26.04) by Theorem C of Phragmén-Lindelöf that $G(z)e^{-\epsilon z}$ is bounded for $x \geqq 0$. But by (26.11) and Theorem C of Phragmén-Lindelöf it follows that

$$|G(z)e^{-\epsilon z}| \leqq 1, \qquad x \geqq 0.$$

Since ϵ is arbitrary it follows that $|G(z)| \leqq 1$ for $x \geqq 0$. Similarly this holds for $x \leqq 0$. Thus $G(z)$ is bounded and must be a constant.

In the general proof we shall give later, the only real change is that $M(u)$ instead of being $10/u^2$ as here will be much larger at $u = 0$. The precise statement of the generalization of Pólya's result is given in the following theorem.

THEOREM XLIV. *Let $G(z)$ be an entire function of zero type; that is,*

(26.12) $$\limsup_{|z|\to\infty} \frac{\log|G(z)|}{|z|} \leqq 0.$$

Let $\{z_n\}$ *be a sequence of complex numbers such that if* $z_n = r_n e^{i\theta_n}$,

$$r_{n+1} - r_n \geqq 1/t(r_n) \tag{26.13}$$

where $t(u) < u^{1/2}$ *is*[3] *a non-decreasing function of* u. *For some* $D > 0$ *let*

$$| n - r_n D | \leqq r_n \theta(r_n) \tag{26.14}$$

where $\theta(u)$ *is a monotone decreasing function such that*

$$\int_1^\infty \frac{\theta(u)}{u} \left\{\log \frac{1}{\theta(4u)} + t(u)\right\} du < \infty, \tag{26.15}$$

and such that $\theta(u)t(u)$ *is monotone decreasing. If*

$$G(\pm z_n) = O(1), \tag{26.16}$$

then $G(z)$ *is a constant.*

In particular if $r_{n+1} - r_n \geqq d$ then (26.15) becomes

$$\int_1^\infty \frac{\theta(u)}{u} \log \frac{1}{\theta(4u)} du < \infty.$$

This condition is satisfied if $\theta(u) = 1/\log^{1+\delta} u$, $\delta > 0$, for large u.

It will be shown in Chapter IX that if

$$\int_1^\infty \frac{\theta(u)}{u} t(u)\, du = \infty$$

then Theorem XLIV does not hold. Thus (26.15) falls short of this necessary condition by the term $\log 1/\theta(4u)$. It is possible to replace (26.16) by

$$G(\lambda_n) = O(1), \qquad G(-\mu_n) = O(1), \qquad n \to \infty,$$

for two positive sequences $\{\lambda_n\}$ and $\{\mu_n\}$ obeying restrictions similar to those on $\{z_n\}$.

The relationship between Theorems XLIII and XLIV is brought out by the following theorem from which Theorem XLIV is derived.

THEOREM XLV. *Let* $G(z)$ *be an entire function of zero type. Let* $\{\lambda_n\}$ *satisfy the inequalities*

$$\liminf_{n\to\infty} \frac{n}{\lambda_n} = D > 0; \qquad \limsup_{n\to\infty} \frac{n}{\lambda_n} < \infty. \tag{26.17}$$

Let

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

Let

[3] $\frac{1}{2}$ is of no special significance here.

(26.18) $$M(u) = \sum_1^\infty \frac{e^{-\lambda_n|u|}}{|F'(\lambda_n)|}.$$

If

$$G(\pm\lambda_n) = O(1)$$

and

$$\int_0^1 \log\log M(u)\,du < \infty,$$

then $G(z)$ is a constant.

We shall show in Chapter IX that this theorem is best possible.

We shall first prove Theorem XLV and use Theorem XLIII in so doing. We require the following lemma.

LEMMA 26.1. *Under the hypothesis of Theorem* XLV *there exists a sequence $\{x_n\}$, $x_n \to \infty$, such that*

(26.19) $$\frac{1}{F(\pm x_n + iy)} = O(e^{Ax_n}), \qquad n \to \infty,$$

for some A and

(26.20) $$|F(x + iy)| \geqq C_0 e^{B|y|}, \qquad |y| > 10|x|,$$

for some $B > 0$ and $C_0 > 0$.

Proof of Lemma 26.1. Clearly

$$|F(x + iy)| \geqq \prod_1^\infty \left|1 - \frac{x^2}{\lambda_n^2}\right| = \left(\prod_{\lambda_n \leqq x/2} \prod_{x/2<\lambda_n<2x} \prod_{\lambda_n \geqq 2x}\right)\left|1 - \frac{x^2}{\lambda_n^2}\right|.$$

Since $1 - u > e^{-10u}$, $u < \frac{1}{4}$, and since

$$\left|\frac{x^2}{\lambda_n^2} - 1\right| > 1, \qquad \lambda_n \leqq \tfrac{1}{2}x,$$

we have

(26.21) $$|F(x + iy)| \geqq \prod_{x/2<\lambda_n<2x} \left|1 - \frac{x^2}{\lambda_n^2}\right| \prod_{\lambda_n \geqq 2x} e^{-10x^2/\lambda_n^2}.$$

But by (26.17) there exists some D_1 such that

(26.22) $$\frac{1}{\lambda_n} \geqq \frac{D_1}{n},$$

and also some D_2 such that

(26.23) $$\frac{1}{\lambda_n} \leqq \frac{D_2}{n}.$$

Thus for large x

$$\prod_{\lambda_n \geqq 2x} e^{-10x^2/\lambda_n^2} \geqq \exp\left\{-\sum_{\lambda_n \geqq 2x} \frac{10x^2 D_2^2}{n^2}\right\}$$

$$\geqq \exp\left\{-10x^2 D_2^2 \sum_{n/D_1 \geqq 2x} \frac{1}{n^2}\right\}$$

$$\geqq \exp\left\{-10x^2 D_2^2 \int_{xD_1}^{\infty} \frac{du}{u^2}\right\} \geqq e^{-10xD_2^2/D_1}.$$

And (26.21) becomes

$$| F(x + iy) | \geqq e^{-10xD_2^2/D_1} \prod_{x/2<\lambda_n<2x} \left| 1 + \frac{x}{\lambda_n} \right| \left| \frac{\lambda_n - x}{\lambda_n} \right|.$$

Taking $x > 0$ this gives

$$| F(x + iy) | \geqq e^{-10xD_2^2/D_1} \prod_{x/2<\lambda_n<2x} \left| \frac{\lambda_n - x}{\lambda_n} \right|. \tag{26.24}$$

Clearly

$$| F'(\lambda_m) | = \frac{2}{\lambda_m} \prod_{n=1}^{\infty}{}' \left| 1 - \frac{\lambda_m^2}{\lambda_n^2} \right|$$

where the prime on the products denotes the omission of the term $n = m$. This becomes, for large m, if $| \lambda_m - x | \leqq 1$,

$$| F'(\lambda_m) | \leqq \prod_{\lambda_n \leqq x/2} \frac{\lambda_m^2}{\lambda_n^2} \prod_{x/2<\lambda_n<2x}{}' \left| 1 - \frac{\lambda_m^2}{\lambda_n^2} \right|.$$

Using (26.22) and (26.23) this becomes

$$| F'(\lambda_m) | \leqq \prod_{n/D_2 \leqq x/2} \frac{\lambda_m^2 D_2^2}{n^2} \prod_{x/2<\lambda_n<2x}{}' \left| 1 - \frac{\lambda_m}{\lambda_n} \right| \prod_{n/D_2<2x} 3$$

$$\leqq \frac{\lambda_m^{xD_2} D_2^{xD_2}}{\Gamma^2(1 + \frac{1}{2}xD_2)} 3^{2xD_2} \prod_{x/2<\lambda_n<2x}{}' \left| \frac{\lambda_n - \lambda_m}{\lambda_n} \right|.$$

Using Stirling's formula there exists some A_1 such that

$$| F'(\lambda_m) | \leqq e^{A_1 x} \prod_{x/2<\lambda_n<2x}{}' \left| \frac{\lambda_n - \lambda_m}{\lambda_n} \right|. \tag{26.25}$$

If $| \lambda_k - x |$ is a minimum for $k = m$, then

$$| \lambda_n - x | \geqq \tfrac{1}{2} | \lambda_n - \lambda_m |,$$

for otherwise $| \lambda_k - x |$ would be a minimum for $k = n$. Thus from (25.24) and (26.25)

$$| F(x + iy) | \geqq e^{-10xD_2^2/D_1 - A_1 x} | F'(\lambda_m) | \left| \frac{\lambda_m - x}{\lambda_m} \right| \prod_{x/2<\lambda_n<2x} \tfrac{1}{2}.$$

Or for some A_2

(26.26) $$|F(x+iy)| \geqq e^{-A_2 x} |F'(\lambda_m)| |\lambda_m - x|.$$

By (26.23) there exist an infinite number of λ_m, $\{\lambda_{m_n}\}$ such that

$$\lambda_{m_n+1} - \lambda_{m_n} \geqq 1/D_2.$$

For such λ_{m_n} let

$$x_n = \lambda_{m_n} + \min \{1, \tfrac{1}{2}(\lambda_{m_n+1} - \lambda_{m_n})\}.$$

Then by (26.26)

(26.27) $$|F(x_n + iy)| \geqq e^{-A_2 x_n} |F'(\lambda_{m_n})| \min (1, 1/2D_2).$$

But by (26.18), since $M(u) < \infty$ for all $u \neq 0$, and in particular for $u = 1$,

$$|F'(\lambda_m)| > e^{-\lambda_m}$$

for all large m. But $\lambda_{m_n} \leqq 2x_n$. Thus (26.27) becomes

$$|F(x_n + iy)| \geqq e^{-(A_2+2)x_n} \min (1, 1/2D_2).$$

Since $F(z)$ is even this gives (26.19).

Clearly

$$\log |F(x+iy)| = \sum_1^\infty \log \left| 1 - \frac{(x+iy)^2}{\lambda_n^2} \right| = \sum_1^\infty \log \left| 1 + \frac{y^2 - x^2 - 2ixy}{\lambda_n^2} \right|.$$

If $|y| > 10|x|$, this gives

$$\log |F(x+iy)| \geqq \sum_1^\infty \log \left(1 + \frac{y^2}{2\lambda_n^2} \right).$$

By (26.22)

$$\log |F(x+iy)| \geqq \sum_1^\infty \log \left(1 + \frac{D_1^2 y^2}{2n^2} \right) = \log \left| \frac{\sin \pi D_1 iy/2^{1/2}}{\pi D_1 y/2^{1/2}} \right|.$$

(26.20) follows easily.

Proof of Theorem XLV. By (26.20) and

(26.28) $$G(z) = O(e^{\epsilon|z|})$$

there exists, for all integers $N > 0$,

(26.29) $$H_N(u) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{G^N(z)}{F(z)} e^{uz}\, dz,$$

and

$$H_N(u+iv) \leqq C \int_{-\infty}^{\infty} e^{-(B-N\epsilon)|y|+|yv|}\, dy,$$

for some C. Since ϵ is arbitrarily small, it follows that $H_N(w)$ exists in the strip $(-\infty < u < \infty, |v| < B)$. Similarly this holds for $H_N'(w)$ and thus $H_N(w)$ is analytic in the strip.

By Cauchy's integral theorem, for large $u > 0$

$$H_N(u) = \frac{1}{2\pi i}\left(\int_{x_n-10ix_n}^{-x_n+10ix_n} + \int_{10ix_n}^{-10ix_n-x_n} + \int_{10ix_n-x_n}^{10ix_n} + \int_{i\infty}^{-10ix_n} + \int_{10ix_n}^{i\infty}\right)\frac{G^N(z)}{F(z)}\, e^{uz}\, dz + \sum_{\lambda_k<x_n} \frac{G^N(-\lambda_k)}{F'(-\lambda_k)}\, e^{-u\lambda_k}.$$

Or, as $n \to \infty$, using (26.19), (26.20) and (26.28),

$$H_N(u) - \sum_{\lambda_k<x_n} \frac{G^N(-\lambda_k)e^{u\lambda_k}}{F'(-\lambda_k)} = O\left(\int_{-10x_n}^{10x_n} e^{11N\epsilon x_n + Ax_n - ux_n}\, dy + \int_0^{x_n} e^{11N\epsilon x_n - 10Bx_n}\, dx + \int_{10x_n}^{\infty} e^{N\epsilon y - By}\, dy\right)$$

$$= O\left(x_n e^{-(u-11N\epsilon-A)x_n} + \frac{x_n}{B - N\epsilon}\, e^{-(10B-11N\epsilon)x_n}\right).$$

Thus for ϵ small and $u > 11N\epsilon + A$, letting $n \to \infty$

$$H_N(u) = \sum_{k=1}^{\infty} \frac{G^N(-\lambda_k)}{F'(-\lambda_k)}\, e^{-u\lambda_k}.$$

By (26.18) and $G(\pm\lambda_k) = O(1)$ it follows that the series above converges for all $u > 0$. Since $H_N(w)$ is analytic it now follows that

$$H_N(w) = \sum_1^{\infty} \frac{G^N(-\lambda_k)}{F'(-\lambda_k)}\, e^{-u\lambda_k}, \qquad u > 0.$$

With no restriction we can assume that $|G(\pm\lambda_n)| \leqq 1$. Thus for $u > 0$

$$(26.30) \qquad |H_N(w)| \leqq \sum_1^{\infty} \frac{e^{-u\lambda_k}}{|F'(-\lambda_k)|} = M(u).$$

Similarly this holds for $u < 0$. From (26.30) it follows that independent of N

$$(26.31) \qquad H_N(u) = O(e^{-(1/2)\lambda_1|u|}), \qquad |u| \to \infty.$$

From (26.30) it is clear that $H_N(u)$ is bounded away from $u = 0$ independent of N. We shall now show that this is also true around $u = 0$.

Since $H_N(w)$ is analytic in the strip $|v| < B$ it certainly is analytic in the square

$$(|u| \leqq \tfrac{1}{2}B, |v| \leqq -\tfrac{1}{2}B).$$

Since in this square

$$|H_N(w)| \leqq M(u),$$

it follows from Theorem XLIII that there exists a constant C_1 independent of N such that

$$| H_N(u) | \leqq C_1, \qquad | u | \leqq \tfrac{1}{2}B.$$

This combined with (26.30) and (26.31) gives

$$| H_N(u) | \leqq C_2 e^{-(1/2)\lambda_1 |u|} \tag{26.32}$$

for some C_2 independent of N.

By the Fourier transform theorem (26.29) gives

$$\frac{G^N(iy)}{F(iy)} = \int_{-\infty}^{\infty} H_N(u) e^{-iuy}\, du.$$

Thus by (26.32)

$$\left| \frac{G^N(iy)}{F(iy)} \right| \leqq C_2 \int_{-\infty}^{\infty} e^{-(1/2)\lambda_1 |u|}\, du \leqq \frac{4C_2}{\lambda_1}.$$

Or

$$| G(iy) | \leqq \left| \frac{4C_2}{\lambda_1} F(iy) \right|^{1/N}.$$

Letting $N \to \infty$, this gives

$$| G(iy) | \leqq 1.$$

By the same Phragmén-Lindelöf argument that follows (26.11) this leads to the fact that $G(z)$ is a constant.

27. Proof of the inequality. We now turn to the proof of Theorem XLIII. We shall use the following lemma.

LEMMA 27.1. *Let $M(x)$ satisfy Theorem* XLIII. *Then there exists a function $\Phi(z)$ analytic except for $z = 0$ such that for any $\delta > 0$ there exists a $C_1(\delta) > 0$ and*

$$| \Phi(z) | \geqq C_1(\delta), \qquad y > \delta. \tag{27.01}$$

Moreover

$$| \Phi(z) | \leqq \frac{1}{M(x)} \tag{27.02}$$

along some cusp symmetrical with respect to the imaginary axis, with vertex at $z = 0$ and opening along the positive imaginary axis, and $\Phi(z)$ is continuous inside and on this cusp.

We shall use this lemma to prove the theorem and then prove the lemma.

Proof of Theorem XLIII. Let $\Phi(z)$ be defined as in Lemma 27.1 and let

$$\Phi_1(z) = \Phi(z + b), \qquad \Phi_2(z) = \Phi(b - z).$$

Let R be the region bounded by two cusps (congruent to the cusp of Lemma 27.1) and two parallel lines as shown in Fig. 3. Let the y coordinate of the

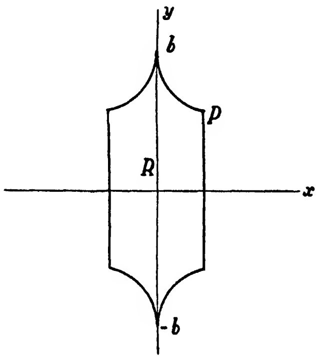

FIG. 3

point P be $b - \delta$, $(\delta > 0)$, and the x coordinate be $x_0 > 0$. Since $\Phi(z)$ is analytic away from $z = 0$

$$| \Phi(z) | < A(\delta), \qquad | z | \geqq \delta,$$

for some $A(\delta) > 0$. By (27.02) on the cusp portions of the boundary of R

$$| \Phi_1(z)\Phi_2(z) | \leqq A(\delta)/M(x),$$

and on the parallel lines portion of the boundary of R

$$| \Phi_1(z)\Phi_2(z) | \leqq A^2(\delta).$$

Recalling that

$$|f(z)| \leqq M(x), \qquad | x | \leqq a, | y | \leqq b,$$

it follows from the above three inequalities that

$$(27.03) \qquad |f(z)\Phi_1(z)\Phi_2(z)| \leqq A(\delta) + A^2(\delta)M(x_0)$$

on the boundary of R. Moreover $f(z)\Phi_1(z)\Phi_2(z)$ is continuous inside and on R and analytic inside of R. Thus by the maximum modulus theorem, (27.03) holds inside of R. Therefore if $| y | \leqq b - \delta$ and $| x | \leqq x_0$,

$$|f(z)| \leqq \frac{A(\delta) + A^2(\delta)M(x_0)}{|\Phi_1(z)\Phi_2(z)|} \leqq \frac{A(\delta) + A^2(\delta)M(x_0)}{C_1^2(\delta)}.$$

Also for $a \geqq x > x_0$, $|f(z)| \leqq M(x_0)$. Thus

$$|f(z)| \leqq \frac{A(\delta) + A^2(\delta)M(x_0)}{C_1^2(\delta)} + M(x_0), \qquad |x| \leqq a, |y| \leqq b - \delta.$$

But this proves Theorem XLIII.

In certain cases the existence of $\Phi(z)$ is easily shown at once. For example if

$$M(x) = e^{e^{1/(2x\log^2 x)}},$$

then it can be easily shown that

$$\Phi(z) = e^{-e^{e^{-1/z^2}}}$$

satisfies the requirements of Lemma 27.1. However to prove Lemma 27.1 for any $M(x)$ is somewhat laborious. We require the following lemma.

LEMMA 27.2. *Given a positive monotone increasing function $N(u)$ such that*

$$N^{3/2}(u) > N'(u) > N^{1/2}(u), \qquad u \to \infty, \tag{27.04}$$

there exists an entire funtion $F(z)$, $(z = x + iy)$, such that

$$\Re F(z) \geqq \exp\left\{\int_1^x N(u)du\right\} \tag{27.05}$$

for

$$|y| \leqq 1/N(x). \tag{27.06}$$

Proof of Lemma 27.2. Since $N(u)$ is increasing and since $N'(u) > N^{1/2}(u)$, for large u, $N'(u) > N^{1/2}(1)$ and therefore $N(u) > uN^{1/2}(1) - $ const. Thus

$$\lim_{u\to\infty} N(u) = \infty.$$

Let $n(u)$ be the inverse function of $N(u)$. Then since $N'(u) > 0$ and $N(u) \to \infty$ as $u \to \infty$, it follows that $n(u)$ is monotone increasing and $n(u) \to \infty$ as $u \to \infty$. Let

$$g(u) = \frac{1}{u}\int_0^u n(v)\, dv. \tag{27.07}$$

Then $g(u) \to \infty$ as $u \to \infty$. Also

$$g'(u) = \frac{n(u)}{u} - \frac{1}{u^2}\int_0^u n(v)\, dv \geqq \frac{n(u)}{u} - \frac{n(u)}{u^2}\int^u dv = 0.$$

Thus $g(u)$ is monotone increasing. Using (27.07) and differentiating $ug(u)$ gives

$$n(u) = g(u) + ug'(u). \tag{27.08}$$

Let

$$F(z) = \int_0^\infty e^{uz - ug(u)}\, du.$$

Since $g(u) \to \infty$ as $u \to \infty$, $F(z)$ is an entire function. Let $z = x + iy$. Let $v = u/N(x)$. Then

$$F(z) = N(x) \int_0^\infty \exp [N(x)\{vz - vg(vN(x))\}]\, dv.$$

Let

$$F_1(z) = \int_0^\infty \exp [N(x)\{vz - vg(vN(x)) - N(x)g'(N(x))\}]\, dv.$$

Then

$$F(z) = F_1(z)N(x) \exp [N^2(x)g'(N(x))]. \tag{27.09}$$

Let $v = 1 + t$. Then

$$F_1(z) = \int_{-1}^\infty \exp [N(x)\{z(1 + t) - (1 + t)g[(1 + t)N(x)] - N(x)g'(N(x))\}]\, dt. \tag{27.10}$$

Let

$$h(t) = -N(x)\{x(1 + t) - (1 + t)g[(1 + t)N(x)] - N(x)g'(N(x))\}. \tag{27.11}$$

Differentiating

$$h'(t) = -N(x)\{x - g[(1 + t)N(x)] - (1 + t)g'[(1 + t)N(x)]N(x)\}.$$

Using (27.08) this becomes

$$h'(t) = -N(x)\{x - n[(1 + t)N(x)]\}. \tag{27.12}$$

Differentiating again

$$h''(t) = N^2(x)n'[(1 + t)N(x)]. \tag{27.13}$$

Using $n(N(x)) = x$ and (27.08),

$$h(0) = 0, \qquad h'(0) = 0.$$

Thus

$$h(t) = \int_0^t (t - v)h''(v)\, dv. \tag{27.14}$$

Using $u = n(v)$, $v = N(u)$ and (27.04),

$$v^{3/2}n'(v) = \frac{N^{3/2}(u)}{N'(u)} > 1, \qquad u \to \infty.$$

Setting $v = (1 + t)N(x)$ it follows for sufficiently large x that

$$(1 + t)^{3/2}N^{3/2}(x)n'[(1 + t)N(x)] > 1, \qquad t \geqq -\tfrac{1}{2}.$$

Using this in (27.13)

(27.15) $$h''(t) > \frac{N^{1/2}(x)}{(1+t)^{3/2}}, \qquad t \geqq -\tfrac{1}{2},$$

for large x. Using this in (27.14),

(27.16) $$\begin{aligned} h(t) &\geqq \int_0^t (t-v)\frac{N^{1/2}(x)}{(1+v)^{3/2}}\,dv \geqq \frac{N^{1/2}(x)}{(1+|t|)^{3/2}}\int_0^t (t-v)\,dv \\ &\geqq \frac{N^{1/2}(x)}{(1+|t|)^{3/2}}\frac{t^2}{2}, \qquad t \geqq -\tfrac{1}{2}. \end{aligned}$$

Since, by (27.13), $h''(t) > 0$ and since $h'(0) = 0$, $h'(t) < 0$ for $t < 0$. Thus

$$h(t) \geqq h(-\tfrac{1}{2}), \qquad t \leqq -\tfrac{1}{2}.$$

Using (27.16) with $t = -\frac{1}{2}$, this gives

(27.17) $$h(t) \geqq \frac{N^{1/2}(x)t^2}{8}, \qquad t \leqq -\tfrac{1}{2}.$$

for large x.

Let

(27.18) $$\tau = N^{-1/8}(x)$$

and let

(27.19) $$I_1 = \left(\int_{-1}^{-\tau} + \int_{\tau}^{1}\right)e^{-h(t)}\,dt.$$

Then by (27.16) and (27.17),

$$\begin{aligned} I_1 &\leqq \left(\int_{-1}^{-\tau} + \int_{\tau}^{1}\right)e^{-N^{1/2}(x)t^2/8}\,dt \\ &= 2\int_{\tau}^{1} e^{-N^{1/2}(x)t^2/8}\,dt \leqq \frac{2}{\tau}\int_{\tau}^{1} e^{-N^{1/2}(x)t^2/8}t\,dt. \end{aligned}$$

Integrating

$$I_1 \leqq \frac{8N^{-1/2}(x)}{\tau}e^{-N^{1/2}\tau^2/8}.$$

Using (27.18) and (27.19)

(27.20) $$\left(\int_{-1}^{-\tau} + \int_{\tau}^{1}\right)e^{-h(t)}\,dt \leqq 8N^{-3/8}(x)e^{-N^{1/4}(x)/8} < \frac{1}{2N^{10}(x)}, \qquad x \to \infty.$$

Again using (27.16)

$$\int_1^\infty e^{-h(t)}\,dt \leqq \int_1^\infty e^{-N^{1/2}(x)t^{1/2}/8}\,dt.$$

Setting $t = s^2$ and integrating by parts

(27.21)
$$\int_1^\infty e^{-h(t)}\,dt \leqq 200e^{-N^{1/2}(x)/8} < \frac{1}{2N^{10}(x)}, \qquad x \to \infty.$$

We now obtain another inequality for $h(t)$. Letting $u = n(v)$, $v = N(u)$, and using (27.04)

$$v^{1/2}n'(v) = \frac{N^{1/2}(u)}{N'(u)} < 1, \qquad u \to \infty.$$

Using $v^{1/2}n'(v) < 1$ in (27.13),

$$h''(t) \leqq \frac{N^2(x)}{(1+t)^{1/2}N^{1/2}(x)}, \qquad t > -\tfrac{1}{2},$$

for large x. Thus for $|t| < \frac{1}{2}$

$$h''(t) \leqq 2N^{3/2}(x).$$

Using this in (27.14)

$$h(t) \leqq t^2N^{3/2}(x), \qquad |t| < \tfrac{1}{2},\ x \to \infty.$$

Therefore with $\tau = N^{-1/8}(x)$

$$\int_{-\tau}^{\tau} e^{-h(t)}\,dt \geqq \int_{-\tau}^{\tau} e^{-t^2N^{3/2}(x)}\,dt.$$

Setting $s = tN^{3/4}(x)$ it follows for large x that

(27.22)
$$\int_{-\tau}^{\tau} e^{-h(t)}\,dt \geqq \frac{1}{N^{3/4}(x)}\int_{-1}^{1} e^{-s^2}\,ds \geqq \frac{1}{10N^{3/4}(x)}.$$

By (27.20) and (27.11)

$$F_1(z) = \int_{-1}^{\infty} e^{iy(t+1)N(x)}\,e^{-h(t)}\,dt.$$

Thus

$$\Re F_1(z) \geqq \int_{-\tau}^{\tau} \cos\{y(t+1)N(x)\}\,e^{-h(t)}\,dt - \left(\int_{-1}^{-\tau} + \int_{\tau}^{\infty}\right)e^{-h(t)}\,dt.$$

Using (27.20) and (27.21)

$$\Re F_1(z) \geqq \int_{-\tau}^{\tau} \cos\{y(t+1)N(x)\}\,e^{-h(t)}\,dt - \frac{1}{N^{10}(x)}.$$

For $|y| \leqq 1/N(x)$ and for large x

$$\cos\{y(t+1)N(x)\} \geqq \cos(1+\tau) > \cos\pi/3 = \tfrac{1}{2}, \qquad |t| \leqq \tau.$$

Thus

$$\Re F_1(z) \geqq \frac{1}{2}\int_{-\tau}^{\tau} e^{-h(t)}\,dt - \frac{1}{N^{10}(x)}.$$

Using (27.22)

$$\Re F_1(z) \geqq \frac{1}{20N^{3/4}(x)} - \frac{1}{N^{10}(x)} > \frac{1}{25N^{3/4}(x)}, \qquad x \to \infty,$$

for $|y| \leqq 1/N(x)$. By (27.09) this gives

(27.23) $$\Re F(z) \geqq \tfrac{1}{25}N^{1/4}(x)\, e^{N^2(x)g'(N(x))}, \qquad |y| \leqq 1/N(x).$$

Since

$$g'(v) = \frac{n(v)}{v} - \frac{1}{v^2}\int_0^v n(t)\,dt,$$

$$g'(N(x)) = \frac{x}{N(x)} - \frac{1}{N^2(x)}\int_0^{N(x)} n(t)\,dt.$$

Setting $n(t) = u$, $t = N(u)$,

$$g'(N(x)) = \frac{x}{N(x)} - \frac{1}{N^2(x)}\int_0^x uN'(u)\,du.$$

Integrating by parts

$$g'(N(x)) = \frac{1}{N^2(x)}\int_0^x N(u)\,du.$$

Using this in (27.23) and recalling that $N(x) \to \infty$ as $x \to \infty$,

$$\Re F(z) \geqq \exp\left\{\int_0^x N(u)\,du\right\}, \qquad |y| \leqq 1/N(x),\ x \to \infty.$$

This completes the proof of the lemma.

Proof of Lemma 27.1. With no restriction we can assume that $\log\log M(x) > 0$. Let

(27.24) $$m(y) = \frac{1}{\log 2}\log\log M(\tfrac{1}{4}y) + \frac{2}{y^{1/2}}.$$

Then $m(y) > 0$ and

(27.25) $$\int_0^1 m(y)\,dy < \infty.$$

Let

(27.26) $$\gamma(u) = \int_0^{1/u} m(y)\,dy, \qquad u > 0.$$

Then $\gamma(u)$ is a monotone decreasing differentiable function of u, and from (27.25)

(27.27) $$\lim_{u\to\infty} \gamma(u) = 0.$$

Let $T(u)$ be the inverse function of $\gamma(u)$. Then $T(u)$ is monotone decreasing

and differentiable and by (27.27)

$$\lim_{u \to +0} T(u) = \infty.$$

Let

$$N(u) = e^{-u} T(e^{-u}).$$

Then

$$\frac{N^{3/2}(u)}{N'(u)} = \frac{e^{-u/2} T^{3/2}(e^{-u})}{-T(e^{-u}) - T'(e^{-u}) e^{-u}}.$$

Setting $e^{-u} = v$

$$\liminf_{u \to \infty} \frac{N^{3/2}(u)}{N'(u)} = \liminf_{v \to +0} \frac{-v^{1/2} T^{3/2}(v)}{vT'(v) + T(v)}.$$

Setting $v = \gamma(u)$, $u = T(v)$, and recalling (27.26), we have

$$\begin{aligned} \liminf_{u \to \infty} \frac{N^{3/2}(u)}{N'(u)} &= \liminf_{u \to \infty} \frac{-\gamma^{1/2}(u) u^{3/2}}{\gamma(u) + u\gamma'(u)} \gamma'(u) \\ &= \liminf_{u \to \infty} \frac{\left(\int_0^{1/u} m(y)\, dy\right)^{1/2} m(1/u) u^{1/2}}{\int_0^{1/u} m(y)\, dy - (1/u) m(1/u)}. \end{aligned} \tag{27.28}$$

Since $m(y)$ is a decreasing function, the denominator above is positive. Thus

$$\begin{aligned} \liminf_{u \to \infty} \frac{N^{3/2}(u)}{N'(u)} &\geqq \liminf_{u \to \infty} \frac{\left(\int_0^{1/u} m(y)\, dy\right)^{1/2} m(1/u) u^{1/2}}{\int_0^{1/u} m(y)\, dy} \\ &= \liminf_{v \to +0} \frac{v^{1/2} m(v)}{\left(\int_0^v m(y)\, dy\right)^{1/2}}. \end{aligned}$$

Since $m(y)$ is integrable $(0, 1)$ it follows that

$$\liminf_{u \to \infty} \frac{N^{3/2}(u)}{N'(u)} \geqq \liminf_{v \to +0} v^{1/2} m(v).$$

From the definition of $m(v)$ this gives

$$\liminf_{u \to \infty} \frac{N^{3/2}(u)}{N'(u)} \geqq 2.$$

This proves part of the inequality (27.04). Incidentally it also follows from (27.28) without taking the limit as $u \to \infty$, that $N'(u) \geqq 0$, $(u > 0)$. The other part of (27.04) we now prove. Just as in (27.28)

$$\begin{aligned} &\limsup_{u \to \infty} \frac{N^{1/2}(u)}{N'(u)} \\ &= \limsup_{u \to \infty} \frac{u^{-3/2} m(1/u)}{\left(\int_0^{1/u} m(y)\, dy\right)^{1/2} \left(\int_0^{1/u} m(y)\, dy - (1/u) m(1/u)\right)}. \end{aligned} \tag{27.29}$$

But by (27.24) since $M(y)$ is decreasing

$$m(y) - m(1/u) \geqq \frac{2}{y^{1/2}} - 2u^{1/2}, \qquad 0 < y < 1/u.$$

Thus

$$\int_0^{1/u} m(y)\,dy - \frac{m(1/u)}{u} = \int_0^{1/u} \{m(y) - m(1/u)\}\,dy$$
$$\geqq \int_0^{1/u} 2\left(\frac{1}{y^{1/2}} - u^{1/2}\right) dy = \frac{2}{u^{1/2}}.$$

Using this in (27.29)

$$\limsup_{u\to\infty} \frac{N^{1/2}(u)}{N'(u)} \leqq \limsup_{u\to\infty} \frac{1}{2u^{1/2}} m(1/u) \Big/ \left(u \int_0^{1/u} m(y)\,dy\right)^{1/2}.$$

Since $m(y)$ is decreasing

$$u \int_0^{1/u} m(y)\,dy \geqq m(1/u).$$

Thus

$$\limsup_{u\to\infty} \frac{N^{1/2}(u)}{N'(u)} \leqq \limsup_{u\to\infty} \frac{1}{2u^{1/2}} m^{1/2}(1/u).$$

Or setting $u = 1/v$,

$$\limsup_{u\to\infty} \frac{N^{1/2}(u)}{N'(u)} \leqq \limsup_{v\to+0} \tfrac{1}{2}\{vm(v)\}^{1/2}. \tag{27.30}$$

But

$$\int_{v/2}^{v} m(y)\,dy \geqq m(v) \int_{v/2}^{v} dy = \tfrac{1}{2}vm(v).$$

Since $m(y)$ is integrable it follows on letting $v \to 0$ that

$$\lim_{v\to+0} vm(v) = 0.$$

Thus (27.30) gives $N^{1/2}(u) \leqq N'(u)$ for large u and this completes the proof of (27.04).

It now follows from Lemma 27.2 that there exists an entire function $F(z)$ such that

$$\Re F(x + iy) \geqq \exp\left\{\int_0^x N(u)\,du\right\}, \qquad |y| \leqq 1/N(x),$$

where $N(u) = e^{-u}T(e^{-u})$. Let

$$f(re^{i\theta}) = F\left(\log \frac{1}{re^{i\theta}}\right).$$

Thus $x + iy = \log 1/r - i\theta$ and

$$\Re f(re^{i\theta}) \geqq \exp\left\{\int_0^{\log 1/r} N(u)\,du\right\}, \qquad |\theta| \leqq \frac{1}{N(\log 1/r)}.$$

But $N(\log 1/r) = rT(r)$. Thus if we set $u = \log 1/\xi$ above,

$$\Re f(re^{i\theta}) \geqq \exp\left\{\int_r^1 T(\xi)\,d\xi\right\}, \qquad |\theta| \leqq 1/rT(r).$$

Let $re^{i\theta} = u + iv$. Then $r = (u^2 + v^2)^{1/2}$ and

$$\Re f(u + iv) \geqq \exp\left\{\int_r^1 T(\xi)\,d\xi\right\}, \qquad |r\theta| \leqq 1/T(r).$$

But for small θ, $|\theta| \leqq 2|\sin\theta|$. Since $r\sin\theta = v$, it follows that

$$\Re f(u + iv) \geqq \exp\left\{\int_r^1 T(\xi)\,d\xi\right\}, \qquad |v| \leqq 1/2T(r),\ u > 0.$$

Since γ is the inverse function of T,

$$\text{(27.31)} \qquad \Re f(u + iv) \geqq \exp\left(\int_{T(r)}^{T(1)} t\gamma'(t)\,dt\right), \qquad r \geqq \gamma(1/2|v|),\ u > 0.$$

Since γ and T are decreasing, if $r = \gamma(1/2|v|)$,

$$\int_{T(r)}^{T(1)} t\gamma'(t)\,dt = -\int_{T(1)}^{T[\gamma(1/2|v|)]} \gamma'(t)t\,dt = -\int_{T(1)}^{1/2|v|} \gamma'(t)t\,dt.$$

But $\gamma'(t) = -m(1/t)/t^2$. Thus

$$\int_{T(r)}^{T(1)} t\gamma'(t)\,dt = \int_{T(1)}^{1/2|v|} m(1/t)\,\frac{dt}{t}, \qquad r = \gamma(1/2|v|),\ u > 0.$$

Setting $t = 1/\xi$ and taking $|v|$ small,

$$\begin{aligned}\int_{T(r)}^{T(1)} t\gamma'(t)\,dt &\geqq \int_{2|v|}^{1/T(1)} m(\xi)\,\frac{d\xi}{\xi} \\ &> \int_{2|v|}^{4|v|} m(\xi)\frac{d\xi}{\xi} \geqq m(4|v|)\int_{2|v|}^{4|v|} \frac{d\xi}{\xi} \\ &= \log 2m(4|v|).\end{aligned}$$

Recalling the definition of $m(u)$,

$$\log 2m(4|v|) > \log\log M(v).$$

Therefore

$$\text{(27.32)} \qquad \int_{T(r)}^{T(1)} t\gamma'(t)\,dt > \log\log M(v), \qquad r = \gamma(1/2|v|),\ u > 0.$$

Thus (27.31) becomes

$$\Re f(u + iv) \geqq \log M(v).$$

Let $z = iw$ and let $\Phi(z) = e^{-f(w)}$. Then by the above inequality

$$(27.33) \qquad |\Phi(z)| \leqq \frac{1}{M(x)}, \qquad |z| = \gamma(1/2|x|),\ y > 0.$$

But $\gamma(u) \to \infty$ as $u \to 0$. Thus

$$|z| = \gamma(1/2|x|), \qquad y > 0$$

is a cusp-shaped curve with vertex at $z = 0$ and opening along the positive imaginary axis, $y > 0$. That $\Phi(z)$ is continuous everywhere except at $z = 0$ follows from the fact that it is analytic everywhere except at $z = 0$ (and at $z = \infty$). By (27.32), since $v \to 0$ as $r \to 0$ it follows that

$$\lim_{r\to 0} \int_{T(r)}^{T(1)} t\gamma'(t)\,dt = \infty.$$

Thus by (27.31)

$$\Re f(u + iv) \to \infty, \qquad r \to 0,\ r \geqq \gamma(1/2|v|),\ u > 0,$$

and therefore $\Phi(z) \to 0$ as $z \to 0$ inside or on the cusp. Thus in and on the cusp $\Phi(z)$ is continuous. This proves (27.02). Since

$$|\Phi(z)| \geqq e^{-|f(-iz)|}$$

and $f(z)$ is analytic everywhere except at $z = 0$ (and $z = \infty$), (27.01) follows. This completes the proof of the lemma.

28. Proof of the theorem on functions of zero type. We shall now prove Theorem XLIV and also prove as a corollary the following theorem.

THEOREM XLVI. *Let $g(z)$ be analytic in the entire plane including infinity except at $z = 1$. Then by modifying $g(z)$ by at most a constant,*

$$(28.01) \qquad g(z) = \sum_0^\infty a_n z^n, \qquad |z| < 1,$$

$$(28.02) \qquad g(z) = -\sum_{-\infty}^{-1} a_n z^n, \qquad |z| > 1.$$

Let $\{m_n\}$ be a subsequence of the positive integers such that, for some $D > 0$,

$$(28.03) \qquad |m_n D - n| \leqq m_n \theta(m_n)$$

where $\theta(u)$ is decreasing and

$$\int_1^\infty \frac{\theta(u)}{u} \log \frac{1}{\theta(4u)}\,du < \infty.$$

If

$$(28.04) \qquad a_{\pm m_n} = O(m_n^K)$$

for some integer K, then

(28.05)
$$g(z) = \sum_1^{K+1} \frac{A_k}{(z-1)^k}.$$

That (28.03) cannot be replaced by

$$\lim_{n\to\infty} \frac{n}{m_n} = D > 0$$

will follow easily from the fact that this cannot be done in Theorem XLIV.

Proof of Theorem XLIV. Let

$$G_1(z) = G(z)G(-z).$$

Then $G_1(z)$ is of zero type and by the Hadamard product theorem, Theorem D, since $G_1(z)$ is even

$$G_1(z) = az^{2k} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\tau_n^2}\right),$$

where k is an integer and $\{\tau_n\}$ is a complex sequence. Let

$$G_2(z) = az^{2k} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{|\tau_n|^2}\right).$$

Then since

$$\left|1 - \frac{r^2}{|\tau_n|^2}\right| \leqq \left|1 - \frac{r^2 e^{2i\theta}}{\tau_n^2}\right|$$

it follows that

$$|G_2(\pm r_n)| \leqq |G_1(\pm z_n)|.$$

Thus by (26.16)

(28.06)
$$G_2(\pm r_n) = O(1), \qquad n \to \infty.$$

Let the number of $|\tau_n| < u$ be $\tau(u)$. Then applying Jensen's theorem, Theorem A, to $G_1(z)$ gives

$$\limsup_{r\to\infty} \frac{1}{r} \int_1^r \frac{\tau(u)}{u}\, du \leqq \frac{1}{2\pi} \int_0^{2\pi} \limsup_{r\to\infty} \frac{\log^+ |G_1(re^{i\theta})|}{r}\, d\theta.$$

Since $G_1(z)$ is of zero type,

$$\lim_{r\to\infty} \frac{1}{r} \int_1^r \frac{\tau(u)}{u}\, du = 0.$$

Clearly

$$\limsup_{|z|\to\infty} \frac{\log |G_2(z)|}{|z|} \leqq \limsup_{|z|\to\infty} \frac{1}{|z|} \sum_1^{\infty} \log\left(1 + \left|\frac{z}{\tau_n}\right|^2\right)$$
$$= \limsup_{r\to\infty} \frac{1}{r} \int_0^{\infty} \log\left(1 + \frac{r^2}{u^2}\right) d\tau(u).$$

Integrating by parts twice

$$\limsup_{|z|\to\infty} \frac{\log |G_2(z)|}{|z|} \leqq \limsup_{r\to\infty} \int_0^\infty \frac{4ur}{(u^2+r^2)^2}\,du \int_1^u \frac{\tau(v)}{v}\,dv$$
$$= \limsup_{r\to\infty} \int_0^\infty \frac{4u^2\,du}{(u^2+1)^2}\,\frac{1}{ur} \int_1^{ur} \frac{\tau(v)}{v}\,dv$$
$$= \limsup_{r\to\infty} \frac{1}{r}\int_1^r \frac{\tau(v)}{v}\,dv \int_0^\infty \frac{4u^2\,du}{(1+u^2)^2}$$
$$= 0.$$

Thus $G_2(z)$ is of zero type

Let

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{r_n^2}\right).$$

Let $\Lambda(u)$ denote the number of $r_n < u$. If a prime on the product or $\Lambda(u)$ denotes the omission of the term r_k, then

$$F'(r_k) = -\frac{2}{r_k} \prod_{n=1}^{\infty}{}' \left(1 - \frac{r_k^2}{r_n^2}\right)$$

or

$$\log |F'(r_k)| = -\log \frac{r_k}{2} + \int_0^\infty \log \left|1 - \frac{r_k^2}{u^2}\right| d\Lambda'(u). \tag{28.07}$$

Since $r_{n+1} - r_n \geqq 1/t(r_n)$,

$$\left|1 - \frac{r_k^2}{r_n^2}\right| \geqq \left|1 - \frac{r_k}{r_n}\right| \geqq \frac{1}{r_n t(r_k + 1)}$$
$$\geqq \frac{1}{2r_k t(2r_k)}, \qquad (r_k - 1 \leqq r_n \leqq r_k + 1),$$

for large k. Also since there are less than $2t(2r_k)$, r_n in $(r_k - 1, r_k + 1)$, (28.07) becomes

$$\log |F'(r_k)| \geqq -\log \frac{r_k}{2} + \left(\int_0^{r_k - 1} + \int_{r_k+1}^\infty\right) \log \left|1 - \frac{r_k^2}{u^2}\right| d\Lambda(u)$$
$$- 2t(2r_k) \log \{2r_k t(2r_k)\}.$$

For $t(u) < u^{1/2}$,

$$\log |F'(r_k)| = \left(\int_0^{r_k-1} + \int_{r_k+1}^\infty\right) \log \left|1 - \frac{r_k^2}{u^2}\right| d\Lambda(u) + O(r_k^{3/4}).$$

Integrating by parts

$$\log |F'(r_k)| = \Lambda(r_k - 1) \log \left|1 - \frac{r_k^2}{(r_k-1)^2}\right| - \Lambda(r_k + 1) \log \left|1 - \frac{r_k^2}{(r_k+1)^2}\right|$$
$$+ \left(\int_0^{r_k-1} + \int_{r_k+1}^\infty\right) \frac{\Lambda(u)}{u}\,\frac{2r_k^2}{r_k^2 - u^2}\,du + O(r_k^{3/4}).$$

But for large k,

$$\Lambda(r_k - 1)\log\left|1 - \frac{r_k^2}{(r_k-1)^2}\right| - \Lambda(r_k+1)\log\left|1 - \frac{r_k^2}{(r_k+1)^2}\right|$$
$$\geqq \Lambda(r_k+1)\left\{\log\left|1 - \frac{r_k^2}{(r_k-1)^2}\right| - \log\left|1 - \frac{r_k^2}{(r_k+1)^2}\right|\right\}$$
$$= \Lambda(r_k+1)\log\frac{2r_k-1}{2r_k+1}\left(\frac{r_k+1}{r_k-1}\right)^2 > 0.$$

Thus for some constant C,

$$(28.08)\qquad \log|F'(r_k)| > -Cr_k^{3/4} + \left(\int_0^{r_k-1} + \int_{r_k+1}^{\infty}\right)\frac{\Lambda(u)}{u}\,\frac{2r_k^2}{r_k^2-u^2}\,du.$$

Setting $v = r_k^2/u$

$$\int_{r_k+1}^{\infty}\frac{\Lambda(u)}{u}\,\frac{2r_k^2}{r_k^2-u^2}\,du = -\int_0^{r_k^2/(r_k+1)}\frac{\Lambda(r_k^2/v)}{r_k^2/v}\,\frac{2r_k^2}{r_k^2-v^2}\,dv.$$

But

$$\frac{r_k^2}{r_k+1} = r_k - 1 + \frac{1}{r_k+1}.$$

Thus (28.08) becomes

$$(28.09)\qquad \begin{aligned}\log|F'(r_k)| > -Cr_k^{3/4} &+ \int_0^{r_k-1}\left(\frac{\Lambda(u)}{u} - \frac{\Lambda(r_k^2/u)}{r_k^2/u}\right)\frac{2r_k^2}{r_k^2-u^2}\,du\\ &- \int_{r_k-1}^{r_k-1+1/(r_k+1)}\frac{\Lambda(r_k^2/u)}{r_k^2/u}\,\frac{2r_k^2}{r_k^2-u^2}\,du.\end{aligned}$$

For large k the last integral above is less than

$$(D+1)\frac{1}{r_k+1}\max_{0<u<r_k-1/2}\frac{2r_k^2}{r_k^2-u^2} < (D+1)\frac{4r_k}{r_k+1} < 4(D+1).$$

Thus if $C_1 = C + 4(D+1)$, (28.09) gives

$$(28.10)\qquad \log|F'(r_k)| \geqq -C_1r_k^{3/4} - \int_0^{r_k-1}\left|\frac{\Lambda(u)}{u} - \frac{\Lambda(r_k^2/u)}{r_k^2/u}\right|\frac{2r_k^2}{r_k^2-u^2}\,du.$$

Clearly (26.14) can be written as

$$|\Lambda(u) - uD| \leqq u\theta(u).$$

Thus for $u < r_k$

$$(28.11)\qquad \left|\frac{\Lambda(u)}{u} - \frac{\Lambda(r_k^2/u)}{r_k^2/u}\right| \leqq \theta(u) + \theta(r_k^2/u) \leqq 2\theta(u).$$

Also if $v > u + 1$,

$$\left|\frac{\Lambda(u)}{u} - \frac{\Lambda(v)}{v}\right| = \left|\frac{(v-u)\Lambda(u) + u\{\Lambda(u) - \Lambda(v)\}}{uv}\right|$$

$$\leqq \frac{\Lambda(u)}{u}\frac{v-u}{v} + \frac{\Lambda(v) - \Lambda(u)}{v}$$

$$\leqq \frac{\Lambda(u)}{u}\frac{v-u}{v} + \frac{(v-u)\{t(v)+1\}}{v}$$

$$\leqq \{2D + t(v) + 1\}\frac{v-u}{v}.$$

Using this and (28.11) in (28.10)

$$(28.12) \qquad \begin{aligned} \log|F'(r_k)| \geqq &-C_1 r_k^{3/4} - 2\int_0^{r_k - r_k\theta(2r_k)} \theta(u)\,\frac{2r_k^2}{r_k^2 - u^2}\,du \\ &- \int_{r_k - r_k\theta(2r_k)}^{r_k - 1} \{2D + t(v) + 1\}\,\frac{(r_k^2/u) - u}{r_k^2/u}\,\frac{2r_k^2}{r_k^2 - u^2}\,du. \end{aligned}$$

If $r_k\theta(2r_k) \leqq 1$ above, then the second integral above does not appear since the range of integration in (28.10) has $r_k - 1$ as its upper limit. Since $\theta(u)$ is decreasing and $t(u)$ non-decreasing, (28.12) gives

$$\begin{aligned} \log|F'(r_k)| \geqq &-C_1 r_k^{3/4} - 8\int_0^{r_k/2}\theta(u)\,du - 4r_k\theta(\tfrac{1}{2}r_k)\int_{r_k/2}^{r_k - r_k\theta(2r_k)} \frac{r_k}{r_k - u}\,du \\ &- 2\int_{r_k - r_k\theta(2r_k)}^{r_k - 1}\{2D + t(2r_k) + 1\}\,du \\ \geqq &-C_1 r_k^{3/4} - 8\int_0^{r_k}\theta(u)\,du - 4r_k\theta(\tfrac{1}{2}r_k)\log\frac{1}{\theta(2r_k)} \\ &- 2\{2D + t(2r_k) + 1\}\,r_k\theta(2r_k). \end{aligned}$$

Or if

$$(28.13) \qquad \begin{aligned} \delta(v) = C_2 v^{-1/4} + \frac{8}{v}\int_0^v \theta(u)\,du + 4\theta(\tfrac{1}{2}v)\log\frac{1}{\theta(2v)} \\ + 2\{2D + t(2v) + 1\}\theta(2v), \end{aligned}$$

then for some C_2

$$(28.14) \qquad \frac{\log|F'(r_k)|}{r_k} \geqq -\delta(r_k), \qquad k \geqq 1.$$

Since $\theta(u)$ and $\theta(u)t(u)$ are decreasing functions, $\delta(u)$ is also decreasing. Let

$$M(u) = \sum_{k=1}^{\infty}\frac{e^{-r_k u}}{|F'(r_k)|}.$$

Then

$$M(u) \leqq \sum_1^\infty e^{r_k\delta(r_k) - ur_k} \leqq \sum_{\delta(r_k)\geq u/2} e^{r_k\delta(r_k)} + \sum_{\delta(r_k)<u/2} e^{-ur_k/2}$$

$$\leqq \sum_{\delta(r_k)\geqq u/2} e^{\delta(r_1)r_k} + \sum_1^\infty e^{-ur_k/2}.$$

Let the inverse function of δ be Δ. Then

$$M(u) \leqq \sum_{r_k \leqq \Delta(u/2)} e^{\delta(r_1)r_k} + \frac{C_3}{u}$$

for some C_3 depending on D. Or

(28.15) $$M(u) \leqq C_4\Delta(\tfrac{1}{2}u)e^{\delta(r_1)\Delta(u/2)} + C_3/u$$

where C_4 also depends on D.

Now by Theorem XLV if we can show that $\log \log M(u)$ is integrable, $(0, 1)$, then it will follow from (28.06) that $G_2(z)$ is a constant. But if $G_2(z)$ is a constant, $G(z)$ must also be a constant. Thus Theorem XLIV will be proved if $\log \log M(u)$ is integrable.

Since, by (28.13), $\delta(v) > C_2v^{-1/4}$, it follows that $\Delta(u) > C_2^4/u^4$. Thus it is clear from (28.15) that $\log \log M(u)$ is integrable over $(0, 1)$ if

$$\int_0^1 \log \Delta(\tfrac{1}{2}u)\, du < \infty,$$

or if

(28.16) $$\int_0^1 \log \Delta(u)\, du < \infty.$$

But if $\Delta(u) = v$, $u = \delta(v)$ and (28.16) is equivalent to

$$-\int_1^\infty \log v\, d\delta(v) < \infty.$$

Integrating by parts this in turn is certainly true if

(28.17) $$\int_1^\infty \frac{\delta(v)}{v}\, dv < \infty.$$

By (26.15) it follows that

(28.18) $$\int_1^\infty \frac{\theta(u)}{u} \log \frac{1}{\theta(4u)}\, du < \infty,$$

and

(28.19) $$\int_1^\infty \frac{\theta(u)t(u)}{u}\, du < \infty.$$

Since $t(u)$ is non-decreasing (28.19) implies

(28.20)
$$\int_1^\infty \frac{\theta(u)}{u}\,du < \infty.$$

Integrating by parts and using (28.20),

$$\int_1^\infty \frac{dv}{v^2}\int_0^v \theta(u)\,du = \left[-\frac{1}{v}\int_0^v \theta(u)\,du\right]_1^\infty + \int_1^\infty \frac{\theta(v)}{v}\,dv < \infty.$$

Using these results in (28.13), the definition of $\delta(v)$, it follows that

(28.21)
$$\int_1^\infty \frac{\delta(v)}{v}\,dv < \infty.$$

This completes the proof of the theorem.

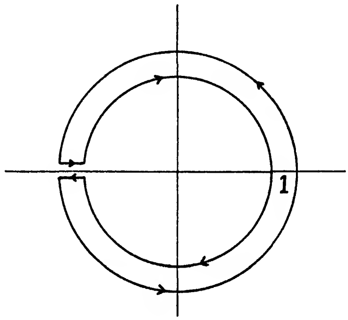

FIG. 4

Proof of Theorem XLVI. Let

(28.22)
$$G(w) = \frac{1}{2\pi i}\int_C \frac{g(z)}{z^{w+1}}\,dz$$

where C is a path in the positive direction around the point $z = 1$ but not enclosing the point $z = 0$. Then $G(w)$ is an entire function. If we take C as a circle of radius $\delta > 0$ around $z = 1$, then

$$|G(w)| \leqq \int_C |g(z)|\,(1+\delta)^{|w|}\,|dz| \leqq e^{\delta|w|}\int_C |g(z)\,dz|.$$

Since this holds for any δ, $G(w)$ is of zero type.[4]

We now deform C into a path as shown in Fig. 4. If w is an integer, positive or negative, and the two parts of the path parallel to the real axis are brought

[4] It is well known that any entire function of $1/(1-z)$ is associated with a function of zero type and conversely. This result appears in the work of Carlson and Wigert.

down to the real axis they cancel each other since z^{n+1} is a one-valued function. Thus if $w = n$ the path breaks up into two parts and if the series (28.01) and (28.02) are used in (28.22),

$$G(n) = -a_n, \qquad -\infty < n < \infty.$$

By (28.04)

$$G(\pm m_n) = O(m_n^K).$$

Now $G(w)$ is of zero type and therefore by Hadamard's factorization theorem, Theorem D, is either a polynomial of degree K or less, or else has more than K zeros. Let us assume that $G(w)$ is not a polynomial of degree K or less. Then if we divide $G(w)$ by K of its zeros and call the resulting entire function of zero type $G_1(w)$,

$$G_1(\pm m_n) = O(1).$$

Thus by (28.03) and Theorem XLIV $G_1(w)$ is a constant. But this means $G(w)$ is a polynomial of degree K. Thus

$$G(w) = B_0 + B_1 w + \cdots + B_K w^K.$$

But $a_n = -G(n)$. Thus

$$a_n = -B_0 - B_1 n - \cdots - B_K n^K.$$

Since

$$g(z) = \sum_0^\infty a_n z^n,$$

it follows that

$$g(z) = \sum_0^{K+1} \frac{A_k}{(z-1)^k}.$$

But by (28.02), $g(z) \to 0$ as $z \to \infty$. Thus $A_0 = 0$. This completes the proof of the theorem.

CHAPTER IX

EXISTENCE OF FUNCTIONS OF ZERO TYPE BOUNDED ON A SEQUENCE OF POINTS

29. Statement of the existence theorems. In this chapter we shall prove that the condition

$$\int_0^1 \log \log M(x)\, dx < \infty \tag{29.01}$$

in Theorems XLIII and XLV is a best possible condition. We shall also give examples of entire functions of zero type, not constants, which are bounded on sequences $\{\pm\lambda_n\}$ of density $D > 0$, and with $\lambda_{n+1} - \lambda_n \geqq d > 0$.

THEOREM XLVII. *Let $M(x)$ be an even function of x decreasing for positive x and $M(x) \to \infty$ as $x \to 0$. Let*

$$\mu(x) = \log \log M(x)$$

be positive and let

$$\int_0^1 \mu(x)\, dx = \infty. \tag{29.02}$$

Let there exist some number $p > 4$ such that

$$\mu(px) < \tfrac{1}{2}\mu(x). \tag{29.03}$$

Then there exists a sequence of functions $\{H_n(z)\}$, $(n > 0)$, analytic for $|y| \leqq a$ for some $a > 0$ such that

$$|H_n(x + iy)| \leqq M(x), \qquad |y| \leqq a, \tag{29.04}$$

and

$$\limsup_{n\to\infty} \max_{|u|\leq 1} |H_n(u)| = \infty. \tag{29.05}$$

Thus for the wide class of $M(x)$ satisfying (29.03), the condition (29.01) is necessary as well as sufficient in Theorem XLIII.

Theorem XLVII is proved by means of the following theorem which proves that Theorem XLV is best possible.

THEOREM XLVIII. *Let $M(x)$ satisfy the requirements of Theorem XLVII. Then there exists a sequence $\{\lambda_n\}$ of density $D > 0$ and with $\lambda_{n+1} - \lambda_n \geqq d > 0$ and an entire function of zero type $G(z)$, not a constant, such that*

$$G(\pm\lambda_n) = O(1); \tag{29.06}$$

and if

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right),$$

then

$$\sum_1^\infty \frac{e^{-\lambda_n |x|}}{|F'(\lambda_n)|} \leqq M(x). \tag{29.07}$$

Here as in the previous case, for the large class of $M(x)$ satisfying (29.03), (29.01) is necessary and sufficient for Theorem XLV.

A corollary of Theorem XLVIII is that there exist functions of zero type, not constants, bounded on a sequence $\{\pm\lambda_n\}$ with $D > 0$ and $\lambda_{n+1} - \lambda_n \geqq d > 0$.

We shall first use Theorem XLVIII to prove Theorem XLVII.

Proof of Theorem XLVII. Since $G(z)$ is not a constant, the sequence of functions

$$H_N(u) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{G^N(z)}{F(z)} e^{uz}\, dz \tag{29.08}$$

defined as in (26.29) cannot be uniformly bounded. For, if they were it would follow, just as in the proof of Theorem XLV, (26.32), that

$$|H_N(u)| \leqq \text{const. } e^{-(1/2)\lambda_1 |u|},$$

and from this that $G(z)$ is a constant. But $H_N(u)$ is uniformly bounded away from zero. For, as in (26.30) (using the Dirichlet series for $H_N(u)$),

$$|H_N(u + iv)| \leqq M(u),$$

for $|v| \leqq a$ for some $a > 0$. Since $M(u)$ is bounded away from zero, $H_N(u)$ is uniformly bounded away from zero. Thus around zero the sequence of functions cannot be bounded. That is,

$$\limsup_{N\to\infty} \max_{|u| \leqq 1} |H_N(u)| = \infty,$$

which is the conclusion of Theorem XLVII.

The remainder of the chapter is devoted to the proof of Theorem XLVIII for which we require several lemmas.

LEMMA 29.1. *Let $\phi(u)$ be a positive increasing, three times differentiable function of u with $\phi'(u)$ decreasing. As $u \to \infty$, let*

$$\phi(u) > u^a, \qquad a > 0, \tag{29.09}$$

$$\liminf_{u\to\infty} \frac{u\phi'(u)}{\phi(u)} > 0, \tag{29.10}$$

$$\phi'(u)\phi(u) = O(1), \qquad u \to \infty, \tag{29.11}$$

$$\phi''(u) = o\{(\phi'(u))^2\}, \phi'''(u) > 0. \tag{29.12}$$

Moreover let

$$\int_1^\infty (\phi'(u))^2\, du = \infty. \tag{29.13}$$

Then there exists an entire function $G(z)$ of zero type and a $\delta > 0$ such that

$$| G(x) | \leqq 1, \quad e^{\phi(n)}(1 - \delta\phi'(n)) \leqq | x | \leqq e^{\phi(n)}(1 + \delta\phi'(n)), \quad n \to \infty. \tag{29.14}$$

To see the significance of this lemma let us take

$$\phi(u) = u^{1/2}$$

which satisfies the requirements of the lemma. In this case $G(x)$ is bounded for

$$e^{n^{1/2}}\left(1 - \frac{\delta}{n^{1/2}}\right) \leqq | x | \leqq e^{n^{1/2}}\left(1 + \frac{\delta}{n^{1/2}}\right).$$

If we take $\{\lambda_n\}$ to be the positive integers which lie in these intervals, it is clear that, if $\Lambda(u)$ is the number $\lambda_n < u$,

$$\Lambda(u) = \sum_{1<n<\log^2 u} \frac{2\delta e^{n^{1/2}}}{n^{1/2}} + O\left(\frac{u}{\log u}\right). \tag{29.15}$$

Since $e^{t^{1/2}}/t^{1/2}$ is an increasing function for $t > 1$

$$\int_1^{(\log u)^2 - 1} \frac{2\delta e^{t^{1/2}}}{t^{1/2}}\, dt < \sum_{1<n<\log^2 u} \frac{2\delta e^{n^{1/2}}}{n^{1/2}} < \int_1^{(\log u)^2 + 1} \frac{2\delta e^{t^{1/2}}}{t^{1/2}}\, dt.$$

Or

$$4\delta\{e^{\log u(1-1/\log^2 u)^{1/2}} - e\} < \sum_{1<n<\log^2 u} \frac{2\delta e^{n^{1/2}}}{n^{1/2}} < 4\delta e^{\log u(1+1/\log^2 u)^{1/2}}.$$

Since

$$\left(1 \pm \frac{1}{\log^2 u}\right)^{1/2} = 1 + O\left(\frac{1}{\log^2 u}\right), \qquad u \to \infty,$$

$$e^{\log u\,(1\pm 1/\log^2 u)^{1/2}} = e^{\log u} e^{O(1/\log u)} = u + O\left(\frac{u}{\log u}\right).$$

Thus

$$\sum_{1<n<\log^2 u} \frac{2\delta e^{n^{1/2}}}{n^{1/2}} = 4\delta u + O\left(\frac{u}{\log u}\right).$$

Using this in (29.15)

$$\Lambda(u) = 4\delta u + O\left(\frac{u}{\log u}\right).$$

This can be written as

$$n - 4\delta\lambda_n = O\left(\frac{\lambda_n}{\log \lambda_n}\right). \tag{29.16}$$

In the notation of Theorem XLIV since λ_n are integers, (29.16) gives

$$\theta(u) = \frac{\text{const.}}{\log u}, \qquad t(u) = 1.$$

Similarly by considering

$$\phi(u) = y^{1/2} \log u, \; \phi(u) = u^{1/2} \log u \log \log u, \text{ etc.}$$

we can get

$$\theta(u) = \text{const.}/\log u \log \log u, \qquad t(u) = 1,$$

$$\theta(u) = \text{const.}/\log u \log \log u \log \log \log u, \qquad t(u) = 1, \text{ etc.}$$

By Lemma 29.1 the conclusion of Theorem XLIV cannot hold for these $\theta(u)$. Thus in the basic criterion, (26.07), of Theorem XLIV, with $t(u) = 1$,

$$\int_1^\infty \frac{\theta(u)}{u} \log \frac{1}{\theta(4u)} \, du < \infty,$$

at most the term $\log 1/\theta(4u)$ is superfluous in the basic criterion of this theorem. We can put Lemma 29.1 with (29.16) and its generalizations into a theorem.

THEOREM XLIX. *There exist sequences of integers $\{\lambda_n\}$ satisfying*

$$| n - D\lambda_n | = O(| \lambda_n | \theta(\lambda_n)), \qquad | n | \to \infty,$$

where $\theta(\lambda_n) = (\log \lambda_n)^{-1}$ or $\theta(\lambda_n) = (\log \lambda_n \log \log \lambda_n)^{-1}$, etc., and functions $G(z)$ of zero type not constants such that

$$G(\pm\lambda_n) = O(1).$$

It is easy to put $\theta(u)$ in a far more general form than that given in Theorem XLIX without modifying Lemma 29.1. It is also possible by modifying Lemma 29.1 slightly to introduce the spacing function $t(u)$ and thereby to show that the basic condition of Theorem XLIV cannot be improved beyond

$$\int_1^\infty \frac{\theta(u)t(u)}{u} \, du < \infty.$$

30. Lemmas. In the remainder of this chapter we shall use $C_1, C_2, \cdots$ for positive constants that are fixed for a given $\phi(u)$. We require a series of lemmas.

LEMMA 30.1. *Let $\phi(u)$ satisfy the requirements of Lemma 29.1. Let $m(u)$ be its inverse function and let*

(30.01) $$\alpha(u) = \int_{1/2}^{u} (\phi'(u))^2 \, du.$$

Let

(30.02) $$A(u) = \frac{e^{\phi(u)} \phi'(u)}{\alpha(u)}.$$

Then

$$\lim_{u\to\infty} \alpha(u) = \infty, \tag{30.03}$$

and, as $u \to \infty$,

$$A'(u) \sim \phi'(u)A(u), \qquad A''(u) > 0, \tag{30.04}$$

$$\frac{d}{du}\{\phi'(u)A(u)\} > 0, \tag{30.05}$$

$$\frac{d}{du}\{\phi'(u)A(u)e^{-2\phi(u)}\} < 0, \tag{30.06}$$

$$\phi'(2u) > C_1\phi'(u), \tag{30.07}$$

$$m(\tfrac{1}{2}u) > C_2 m(u), \tag{30.08}$$

$$\alpha(\tfrac{3}{4}u) > \tfrac{1}{2}\alpha(u), \tag{30.09}$$

and

$$A(u) > e^{(3/4)\phi(u)}. \tag{30.10}$$

Proof of Lemma 30.1. From (29.13) and (30.01), (30.03) follows. Differentiating

$$\log A(u) = \phi(u) + \log \phi'(u) - \log \alpha(u)$$

gives

$$\frac{A'(u)}{A(u)} = \phi'(u) + \frac{\phi''(u)}{\phi'(u)} - \frac{(\phi'(u))^2}{\alpha(u)}, \tag{30.11}$$

and again differentiating

$$\begin{aligned}\frac{A''(u)A(u) - (A'(u))^2}{A^2(u)} &= \phi''(u) + \frac{\phi'(u)\phi'''(u) - (\phi''(u))^2}{(\phi'(u))^2} \\ &\quad - \frac{2\phi'(u)\phi''(u)}{\alpha(u)} + \frac{(\phi'(u))^4}{\alpha^2(u)}.\end{aligned} \tag{30.12}$$

Obviously

$$\frac{A''(u)}{A(u)} = \frac{A''A(u) - (A'(u))^2}{A^2(u)} + \left(\frac{A'(u)}{A(u)}\right)^2.$$

Using (30.11) and (30.12) in the above,

$$\begin{aligned}\frac{A''(u)}{A(u)} &= (\phi'(u))^2 + \frac{2(\phi'(u))^4}{\alpha^2(u)} + 3\phi''(u) - \frac{2(\phi'(u))^3}{\alpha(u)} \\ &\quad - \frac{4\phi'(u)\phi''(u)}{\alpha(u)} + \frac{\phi'''(u)}{\phi'(u)}.\end{aligned} \tag{30.13}$$

Using $\alpha(u) \to \infty$, $\phi'(u) = O(1)$, $\phi''(u) = o\{(\phi'(u))^2\}$, and $\phi'''(u) > 0$ in (30.11) and (30.13) gives (30.01). We also have

$$\frac{d}{du}\{\phi'(u)A(u)\} = \phi'(u)A'(u) + A(u)\phi''(u)$$
$$= A(u)\left\{\phi'(u)\frac{A'(u)}{A(u)} + \phi''(u)\right\}.$$

Using (30.11) this becomes

$$\frac{d}{du}\{\phi'(u)A(u)\} = A(u)\left\{(\phi'(u))^2 + 2\phi''(u) - \frac{(\phi'(u))^3}{\alpha(u)}\right\},$$

and from the behavior of α, ϕ', and ϕ'' as $u \to \infty$ (30.05) follows. (30.06) is proved in the same way.

By (29.10) there exists an $\epsilon > 0$ such that for large u

$$\frac{u\phi'(u)}{\phi(u)} > 2\epsilon. \tag{30.14}$$

Clearly

$$\phi(u) \geqq \int_0^{u/2} \phi'(y)\,dy.$$

Since $\phi'(y)$ is decreasing, it follows that

$$\phi(u) > \tfrac{1}{2}u\phi'(\tfrac{1}{2}u).$$

With (30.14) this gives

$$\frac{1}{2\epsilon}u\phi'(u) > \tfrac{1}{2}u\phi'(\tfrac{1}{2}u).$$

This gives (30.07) with $C_1 = \epsilon$.

Next if $v = \phi(u)$ then $u = m(v)$. Thus (30.14) becomes

$$\frac{m(v)}{vm'(v)} > 2\epsilon. \tag{30.15}$$

But

$$m(v) = m((1 - \epsilon)v) + \int_{(1-\epsilon)v}^{v} m'(y)\,dy.$$

Since $m'(v)$ is the reciprocal of $\phi'(u)$, $m'(v)$ is an increasing function. Thus

$$m(x) \leqq m((1 - \epsilon)v) + \epsilon vm'(v).$$

But by (30.15) this gives $m(v) \leqq m((1 - \epsilon)v) + \frac{1}{2}m(v)$. Or

$$\tfrac{1}{2}m(v) \leqq m((1 - \epsilon)v).$$

Iterating this we obtain (30.08).

Since $\phi'(x)$ is positive and decreasing,

$$\alpha(u) = \int_{1/2}^{u} (\phi'(y))^2\,dy \leqq 2\int_{1/2}^{u/2+1/2} (\phi'(y))^2\,dy = 2\alpha(\tfrac{1}{2}u + \tfrac{1}{2}), \qquad u > 2,$$

which gives (30.09).

Finally

$$\alpha(u) = \int_{1/2}^{u} (\phi'(y))^2\,dy \leqq \phi'(1)\int_{1/2}^{u} \phi'(y)\,dy \leqq \phi'(1)\phi(u).$$

Thus

$$A(u) \geqq \frac{e^{\phi(u)}\phi'(u)}{\phi'(1)\phi(u)}.$$

Using (30.14) this becomes

$$A(u) \geqq \frac{\epsilon e^{\phi(u)}}{u} \geqq \epsilon e^{(3/4)\phi(u)}\,\frac{e^{\phi(u)/4}}{u}.$$

From (29.09), (30.10) now follows. This completes the proof of the lemma.

LEMMA 30.2. *Let N be a positive integer and let*

(30.16) $$B(u) = \log|1 - e^{2\phi(N)+2y-2\phi(u)}|$$

where

(30.17) $$|y| < \tfrac{1}{2}\phi'(N+1) < \tfrac{1}{2}\phi'(N).$$

Then

(30.18) $$\sum_{1}^{N-1} A(n)B(n) = \tfrac{1}{2}A(N-1)B(N-1) + \int_{1}^{N-1} A(u)B(u)\,du + O(A(N)),$$

and

(30.19) $$\sum_{N+1}^{\infty} A(n)B(n) = \tfrac{1}{2}A(N+1)B(N+1) + \int_{N+1}^{\infty} A(u)B(u)\,du + O(A(N)).$$

Proof of Lemma 30.2. Let

(30.20) $$P(u) = [u] - u + \tfrac{1}{2},$$

where $[u]$ is the largest integer not exceeding u. Then by the Euler summation formula

(30.21) $$\sum_{1}^{N-1} A(n)B(n) = \tfrac{1}{2}A(N-1)B(N-1) + \tfrac{1}{2}A(1)B(1) + \int_{1}^{N-1} A(u)B(u)\,du - \int_{1}^{N-1} P(u)\frac{d}{du}\{A(u)B(u)\}\,du.$$

Clearly from the periodicity of $P(u)$, for any a and b

$$\left|\int_a^b P(u)\,du\right| \leqq \tfrac{1}{8}. \tag{30.22}$$

Let $T(u)$ be a differentiable function. Then integrating by parts

$$\int_1^{N-1} P(u)T(u)\,du = T(N-1)\int_1^{N-1} P(u)\,du - \int_1^{N-1} T'(u)\,du \int_1^u P(v)\,dv.$$

By (30.22) this becomes

$$\left|\int_1^{N-1} P(u)T(u)\,du\right| \leqq \tfrac{1}{8}|\,T(N-1)\,| + \tfrac{1}{8}\int_1^{N-1} |\,T'(u)\,|\,du,$$

or

$$\left|\int_1^{N-1} P(u)T(u)\,du\right| \leqq \tfrac{1}{4}\int_1^{N-1} |\,T'(u)\,|\,du + \tfrac{1}{8}|\,T(1)\,| \tag{30.23}$$

We now consider

$$\frac{d}{du}\{A(u)B(u)\} = A'(u)B(u) + A(u)B'(u). \tag{30.24}$$

By (30.16)

$$A(u)B'(u) = A(u)\,\frac{2\phi'(u)e^{2y+2\phi(N)-2\phi(u)}}{1 - e^{2y+2\phi(N)-2\phi(u)}}\,.$$

Or

$$A(u)B'(u) = -A(u)\phi'(u)\left(\frac{2e^{2y+2\phi(N)}}{e^{2\phi(N)+2y} - e^{2\phi(u)}}\right).$$

By (30.05), $A(u)\phi'(u)$ is an increasing function of u for large u and the other term in the above expression for $A(u)B'(u)$ is obviously increasing for $1 \leqq u \leqq N-1$. Thus $A(u)B'(u)$ is a negative decreasing function for $C_3 < u < N-1$, and therefore

$$\begin{aligned}\int_1^{N-1}\left|\frac{d}{du}\{A(u)B'(u)\}\right|du &\leqq \left|\int_{C_3}^{N-1}\frac{d}{du}\{A(u)B'(u)\}\,du\right| \\ &\qquad + \int_1^{C_3}\left|\frac{d}{du}\{A(u)B'(u)\}\right|du \\ &\leqq |\,A(N-1)B'(N-1)\,| + C_4.\end{aligned}$$

Thus by (30.23) with $T(u) = A(u)B'(u)$,

$$\left|\int_1^{N-1} P(u)A(u)B'(u)\,du\right| \leqq \tfrac{1}{4}|\,A(N-1)B'(N-1)\,| + C_5. \tag{30.25}$$

But

$$B'(N-1) = -2\phi'(N-1)\,\frac{e^{2y+2\phi(N)-2\phi(N-1)}}{e^{2y+2\phi(N)-2\phi(N-1)} - 1}\,.$$

Since $\phi'(u)$ is decreasing and $|y| < \frac{1}{2}\phi'(N)$,

(30.26) $$y + \phi(N) - \phi(N-1) = \int_{N-1}^{N} \phi'(u)\,du + y \geqq \phi'(N) - \tfrac{1}{2}\phi'(N) = \tfrac{1}{2}\phi'(N).$$

Thus

$$|B'(N-1)| \leqq 2\phi'(N-1)\frac{e^{\phi'(N)}}{e^{\phi'(N)} - 1} < \frac{2\phi'(N-1)}{\phi'(N)}e^{\phi'(N)}.$$

But by (30.07), $\phi'(N-1) < C_6\phi'(N)$. Thus

$$|B'(N-1)| < 2C_6e^{\phi'(N)}.$$

Since $\phi'(N)$ is a positive decreasing function, $B'(N-1)$ is bounded and (30.25) becomes

(30.27) $$\int_1^{N-1} P(u)A(u)B'(u)\,du = O(A(N-1)) = O(A(N)).$$

The other term in (30.24), $A'(u)B(u)$ has as its derivative

$$A''(u)\log|1 - e^{2y+2\phi(N)-2\phi(u)}| + A'(u)\frac{2\phi'(u)e^{2y+2\phi(N)-2\phi(u)}}{1 - e^{2y+2\phi(N)-2\phi(u)}}.$$

Since $A''(u) > 0$ for large u, and since the logarithmic term above is steadily decreasing from 1 to $N-1$, taking extreme values in magnitude at 1 and $N-1$,

$$\int_1^{N-1} |A''(u)B(u)|\,du \leqq (|B(1)| + |B(N-1)|)\left(\int_1^{N-1} A''(u)\,du + C_7\right)$$
$$\leqq (|B(1)| + |B(N-1)|)(A'(N-1) + C_8).$$

But for large N, $|B(1)| < 3\phi(N)$ and by (30.26)

$$|B(N-1)| \leqq \log\frac{10}{\phi'(N)}.$$

Thus

$$\int_1^{N-1} |A''(u)B(u)|\,du = O\left(\left\{\phi(N) + \log\frac{1}{\phi'(N)}\right\}A'(N)\right).$$

But $A'(N) \sim \phi'(N)A(N)$ and $\phi(N)\phi'(N) = O(1)$. Thus

$$\int_1^{N-1} |A''(u)B(u)|\,du = O\left(A(N)\left\{1 + \phi'(N)\log\frac{1}{\phi'(N)}\right\}\right).$$

Since $\phi'(N)$ is decreasing, it follows that

(30.28) $$\int_1^{N-1} |A''(u)B(u)|\,du = O(A(N)).$$

For the other term in the derivative of $A'(u)B(u)$ we have, since $A'(u)$ is

positive and increasing for large u,

$$\int_1^{N-1} |A'(u)B'(u)|\,du = \int_1^{N-1} \left| A'(u)\,\frac{2\phi'(u)e^{2y+2\phi(N)-2\phi(u)}}{e^{2y+2\phi(N)-2\phi(u)}-1}\right| du$$

$$\leqq A'(N)\int_1^{N-1} \frac{2\phi'(u)e^{2y+2\phi(N)-2\phi(u)}}{e^{2y+2\phi(N)-2\phi(u)}-1}\,du + C_9$$

$$= -A'(N)\log\left(e^{2y+2\phi(N)-2\phi(u)}-1\right)\Big]_1^{N-1} + C_9.$$

Handling the logarithmic term at 1 and $N-1$ and $A'(N)$ in exactly the same way as in obtaining (30.28) we have

$$\int_1^{N-1} |A'(u)B'(u)|\,du = O(A(N)).$$

Combining this with (30.28),

$$\int_1^{N-1} \left|\frac{d}{du}\{A'(u)B(u)\}\right| du = O(A(N)).$$

Setting $T(u) = A'(u)B(u)$ in (30.23) this gives

$$\left|\int_1^{N-1} P(u)A'(u)B(u)\,du\right| \leqq C_{10}A(N) + |A'(1)B(1)|. \tag{30.29}$$

But $B(1) = O(\phi(N))$. Since

$$A(N) > e^{(3/4)\phi(N)} > \phi(N)$$

for large N, (30.29) becomes

$$\int_1^{N-1} P(u)A'(u)B(u)\,du = O(A(N)). \tag{30.30}$$

Using (30.27) and (30.30) in (30.21), we obtain

$$\sum_1^{N-1} A(n)B(n) = \tfrac{1}{2}A(N-1)B(N-1) + \int_1^{N-1} A(u)B(u)\,du + O(A(N)),$$

completing one-half of the lemma.

Again by the Euler summation formula,

$$\begin{aligned}\sum_{N+1}^{\infty} A(n)B(n) &= \tfrac{1}{2}A(N+1)B(N+1) + \int_{N+1}^{\infty} A(u)B(u)\,du \\ &\quad - \int_{N+1}^{\infty} P(u)\frac{d}{du}\{A(u)B(u)\}\,du.\end{aligned} \tag{30.31}$$

As in (30.23) if $T(u)$ is any differentiable function such that $T(u) \to 0$ as $u \to \infty$,

$$\left|\int_{N+1}^{\infty} P(u)T(u)\,du\right| \leqq \tfrac{1}{2}\int_{N+1}^{\infty} |T'(u)|\,du. \tag{30.32}$$

Again we consider the two terms in

$$\frac{d}{du}\{A(u)B(u)\}$$

separately. $A(u)B'(u)$ is now best written in the form

$$A(u)B'(u) = 2\phi'(u)A(u)e^{-2\phi(u)}\frac{e^{2y+2\phi(N)}}{1 - e^{2y+2\phi(N)-2\phi(u)}}.$$

But by (30.06) $\phi'(u)A(u)e^{-2\phi(u)}$ is decreasing for large u. The term

$$\frac{1}{1 - e^{2y+2\phi(N)-2\phi(u)}}$$

for $u \geqq N + 1$ is decreasing. Thus $A(u)B'(u)$ is a decreasing function for $u \geqq N + 1$ and therefore its derivative has one sign. Thus by (30.32), for large N

$$\left|\int_{N+1}^{\infty} P(u)A(u)B'(u)\,du\right| \leqq -\tfrac{1}{4}\int_{N+1}^{\infty}\frac{d}{du}\{A(u)B'(u)\}\,du$$
$$= \tfrac{1}{4}A(N+1)B'(N+1).$$

As in the results following (30.25), $B'(N + 1)$ is bounded. Thus

$$\int_{N+1}^{\infty} P(u)A(u)B'(u)\,du = O(A(N+1)). \tag{30.33}$$

The derivative of the other part of $(d/du)\{A(u)B(u)\}$ is

$$\frac{d}{du}\{A'(u)B(u)\} = A''(u)B(u) + A'(u)B'(u). \tag{30.34}$$

Integrating by parts $A''(u)B(u)$ gives

$$\left|\int_{N+1}^{\infty} A''(u)B(u)\,du\right| \leqq |A'(N+1)B(N+1)| + \left|\int_{N+1}^{\infty} A'(u)B'(u)\,du\right|.$$

Since for large u, $A''(u) > 0$, and $B(u) < 0$ for $u > N + 1$,

$$\int_{N+1}^{\infty} |A''(u)B(u)|\,du = \left|\int_{N+1}^{\infty} A''(u)B(u)\,du\right|.$$

Thus

$$\int_{N+1}^{\infty} |A''(u)B(u)|\,du \leqq |A'(N+1)B(N+1)| + \left|\int_{N+1}^{\infty} A'(u)B'(u)\,du\right|. \tag{30.35}$$

But

$$B(N+1) = \log|1 - e^{2y+2\phi(N)-2\phi(N+1)}|,$$

and much as in (30.26) $y + \phi(N) - \phi(N+1) < -\tfrac{1}{2}\phi'(N+1)$. Thus

$$|B(N+1)| \leqq \log \frac{10}{\phi'(N+1)}.$$

Also $A'(N+1) \sim \phi'(N+1)A(N+1)$. Thus

$$|A'(N+1)B(N+1)| \leqq 10A(N+1)\phi'(N+1) \log \frac{10}{\phi'(N+1)}.$$

Since $\phi'(u)$ is decreasing, this becomes for large N

$$|A'(N+1)B(N+1)| \leqq C_{11}A(N+1).$$

Thus (30.35) becomes for large N

$$\text{(30.36)} \qquad \int_{N+1}^{\infty} |A''(u)B(u)|\,du \leqq C_{11}A(N+1) + \int_{N+1}^{\infty} |A'(u)B'(u)|\,du.$$

The other term of (30.34) is handled in the following way. Since $A'(u) \sim (\phi\,(u))^2 e^{\phi(u)}/\alpha(u)$,

$$\int_{N+1}^{\infty} |A'(u)B'(u)|\,du \leqq 2\int_{N+1}^{\infty} \frac{(\phi'(u))^2 e^{\phi(u)}}{\alpha(u)} \frac{2\phi'(u)e^{2y+2\phi(N)-2\phi(u)}}{1-e^{2y+2\phi(N)-2\phi(u)}}\,du$$

$$\leqq \frac{4(\phi'(N+1))^2}{\alpha(N+1)} \int_{N+1}^{\infty} \frac{e^{2y+2\phi(N)-\phi(u)}}{1-e^{2y+2\phi(N)-2\phi(u)}}\,\phi'(u)\,du$$

Let $x = e^{y+\phi(N)-\phi(u)}$. Then

$$\int_{N+1}^{\infty} |A'(u)B'(u)|\,du \leqq \frac{4(\phi'(N+1))^2}{\alpha(N+1)} e^{y+\phi(N)} \int_0^{e^{y+\phi(N)-\phi(N+1)}} \frac{dx}{1-x^2}$$

$$\leqq \frac{4(\phi'(N+1))^2}{\alpha(N+1)} e^{y+\phi(N)} \log \frac{1+e^{y+\phi(N)-\phi(N+1)}}{1-e^{y+\phi(N)-\phi(N+1)}}.$$

But $\phi(N+1) - \phi(N) - y \geqq \frac{1}{2}\phi'(N+1)$. Thus

$$\int_{N+1}^{\infty} |A'(u)B'(u)|\,du = O\left(\frac{(\phi'(N+1))^2}{\alpha(N+1)} e^{\phi(N)} \log \frac{1}{\phi'(N+1)}\right)$$

$$= O\left(\frac{\phi'(N)}{\alpha(N)} e^{\phi(N)}\right) = O(A(N)).$$

Using this and (30.36) with (30.34) gives

$$\text{(30.37)} \qquad \int_{N+1}^{\infty} \left|\frac{d}{du}\{A'(u)B(u)\}\right| du = O(A(N+1)).$$

But since $A'(y) \sim \phi'(y)A(y)$,

$$A(N+1) = A(N) + \int_N^{N+1} A'(y)\,dy \leqq A(N) + 2\int_N^{N+1} \phi'(y)A(y)\,dy.$$

Since

$$\phi'(u) = O\left(\frac{1}{\phi(u)}\right),$$

$$A(N+1) \leqq A(N) + C_{12}\frac{A(N+1)}{\phi(N)}.$$

Or

$$A(N+1)\left(1 - \frac{C_{13}}{\phi(N)}\right) \leqq A(N).$$

Thus for large N

$$A(N+1) = A(N) + O\left(\frac{A(N)}{\phi(N)}\right). \tag{30.38}$$

Thus (30.37) becomes

$$\int_{N+1}^{\infty}\left|\frac{d}{du}\{A'(u)B(u)\}\right| du = O(A(N)).$$

Using this in (30.32) with $T(u) = A'(u)B(u)$,

$$\int_{N+1}^{\infty} P(u)A'(u)B(u)\,du = O(A(N)).$$

Combining this with (30.33) gives

$$\int_{N+1}^{\infty} P(u)\frac{d}{du}\{A(u)B(u)\}\,du = O(A(N)).$$

Using this in (30.31) completes the proof of the lemma.

Proof of Lemma 29.1. Let

$$G(z) = \prod_{1}^{\infty}\left(1 - \frac{z^2}{e^{2\phi(n)}}\right)^{[A(n)]+1}$$

where $A(n)$ is defined in (30.02) and $[A(n)]$ is the largest integer not exceeding $A(n)$. Then

$$\log|G(x)| = \sum_{1}^{\infty}([A(n)]+1)\log|1 - x^2e^{-2\phi(n)}|.$$

Replacing $[A(n)]+1$ by a smaller quantity when it is the coefficient of a logarithmic term that is negative and by a larger quantity when it is the coefficient of a positive term,

$$\log|G(x)| \leqq \sum_{1}^{\infty}A(n)\log|1 - x^2e^{-2\phi(n)}| + \sum_{\phi(n)<\log 2^{-1/2}x}\log|x^2e^{-2\phi(n)} - 1|$$

for $x > 10$. But

$$\sum_{\phi(n)<\log 2^{-1/2}x}\log|x^2e^{-2\phi(n)} - 1| < \sum_{\phi(n)<\log x}\log x^2 < 2\log x\sum_{\phi(n)<\log x}1.$$

But $\phi(n) > n^a$ for large n. Thus

$$\sum_{\phi(n)<\log x} 1 = O(\sum_{n^a<\log x} 1) = O(\log^{1/a} x).$$

Combining the three inequalities above,

$$\log |G(x)| \leqq \sum_1^\infty A(n) \log |1 - x^2 e^{-2\phi(n)}| + C_{13} \log^{1/a} x.$$

Let

$$x = e^{\phi(N)+y}, \qquad |y| < \tfrac{1}{2}\phi'(N+1), \tag{30.39}$$

where y is unrelated to $\Im z$. Recalling the definition of $B(u)$,

$$\log|G(x)| \leqq \left(\sum_1^{N-1} + \sum_{N+1}^\infty\right) A(n)B(n) + A(N)\log|1 - e^{2y}| + C_{13}|\log x|^{1/a}. \tag{30.40}$$

But by (30.39),

$$(\log x)^{1/a} = O((\phi(N))^{1/a}) = O(e^{(3/4)\phi(N)}) = O(A(N)).$$

Using this, (30.18) and (30.19) in (30.40),

$$\begin{aligned} \log |G(x)| \leqq\ & A(N) \log |1 - e^{2y}| + \tfrac{1}{2}A(N-1)B(N-1) \\ & + \tfrac{1}{2}A(N+1)B(N+1) \\ & + \left(\int_1^{N-1} + \int_{N+1}^\infty\right) A(u)B(u)\,du + C_{14}A(N). \end{aligned} \tag{30.41}$$

We now consider

$$\int_1^{N-1} A(u)B(u)\,du = \int_1^{N-1} \frac{e^{\phi(u)}\phi'(u)}{\alpha(u)} \log |x^2 e^{-2\phi(u)} - 1|\,du. \tag{30.42}$$

Since $m(u)$ is the inverse function of $\phi(u)$, if $v = e^{\phi(u)}$, the equation (30.42) becomes

$$\int_1^{N-1} A(u)B(u)\,du = \int_{e^{\phi(1)}}^{e^{\phi(N-1)}} \frac{dv}{\alpha(m(\log v))} \log\left(\frac{x^2}{v^2} - 1\right).$$

Let $v = xt$. Then

$$\begin{aligned} \int_1^{N-1} A(u)B(u)\,du &= x\int_{e^{\phi(1)}/x}^{e^{\phi(N-1)}/x} \frac{dt}{\alpha(m(\log xt))} \log\left(\frac{1}{t^2} - 1\right) \\ &= x\int_{1/x^{1/2}}^{e^{\phi(N-1)}/x} \frac{dt}{\alpha(m(\log xt))} \log\left(\frac{1}{t^2} - 1\right) + O(x^{3/4}). \end{aligned} \tag{30.43}$$

But for $t > 1/x^{1/2}$, by the mean value theorem

$$(30.44)\quad \frac{1}{\alpha\{m(\log x + \log t)\}} = \frac{1}{\alpha\{m(\log x)\}} - \frac{\alpha'\{m(\log x + h\log t)\}m'(\log x + h\log t)}{\alpha^2\{m(\log x + h\log t)\}}\log t$$

where $0 < h < 1$. Since $\alpha' = (\phi')^2$ and $m'(u) = 1/\phi'(m(u))$, (30.44) becomes

$$(30.45)\quad \left|\frac{1}{\alpha\{m(\log x + \log t)\}} - \frac{1}{\alpha\{m(\log x)\}}\right| \leqq \frac{\phi'\{m(\log x + h\log t)\}\,|\log t|}{\alpha^2\{m(\log x + h\log t)\}}.$$

Since $m(u)$ is monotone increasing and $t > 1/x^{1/2}$,

$$m(\log x + h\log t) > m(\log x - \log x^{1/2}) \geqq m(\tfrac{1}{2}\log x).$$

By (30.08), for large N this gives

$$\begin{aligned} m(\log x + h\log t) &\geqq C_2 m(\log x) = C_2 m(\phi(N) + y) \\ &\geqq C_2 m(\phi(N) - \phi'(N)) \geqq C_2 m(\phi(N-1)) \\ &= C_2(N-1) > C_2^2 N. \end{aligned}$$

Thus since α is increasing and ϕ' decreasing (30.45) becomes

$$(30.46)\quad \left|\frac{1}{\alpha\{m(\log x + \log t)\}} - \frac{1}{\alpha\{m(\log x)\}}\right| \leqq \frac{\phi'(C_2^2 N)\,|\log t|}{\alpha^2(C_2^2 N)}.$$

Using this in (30.43),

$$(30.47)\quad \begin{aligned} \int_1^{N-1} A(u)B(u)\,du &= \frac{x}{\alpha(m(\log x))}\int_{1/x^{1/2}}^{e^{\phi(N-1)}/x} \log\left(\frac{1}{t^2} - 1\right)dt \\ &\quad + O\left(\frac{\phi'(C_2^2N)x}{\alpha^2(C_2^2N)}\int_0^1 \log\frac{1}{t}\left|\log\left(\frac{1}{t^2} - 1\right)\right|dt + x^{3/4}\right) \\ &= \frac{x}{\alpha(m(\log x))}\int_0^{e^{\phi(N-1)}/x} \log\left(\frac{1}{t^2} - 1\right)dt \\ &\quad + O\left(x^{3/4} + x\frac{\phi'(C_2^2N)}{\alpha^2(C_2^2N)}\right). \end{aligned}$$

Making the transformation $t = e^{\phi(u)}/x$ and using (30.46) as above,

$$(30.48)\quad \begin{aligned} \int_{N+1}^{\infty} A(u)B(u)\,du &= \frac{x}{\alpha(m(\log x))}\int_{e^{\phi(N+1)}/x}^{\infty} \log\left(1 - \frac{1}{t^2}\right)dt \\ &\quad + O\left(\frac{x\phi'(C_2^2N)}{\alpha^2(C_2^2N)}\right). \end{aligned}$$

By iterating (30.07) and (30.09),

$$\phi'(C_2^2N) = O(\phi'(N)), \qquad \alpha(N) = O(\alpha(C_2^2N)).$$

Thus

$$\frac{x\phi'(C_2^2 N)}{\alpha^2(C_2^2 N)} = O\left(\frac{x\phi'(N)}{\alpha^2(N)}\right) = O\left(\frac{e^{\phi(N)}\phi'(N)}{\alpha(N)}\right) = O(A(N)).$$

By (30.10)

$$x^{3/4} = O(e^{(3/4)\phi(N)}) = O(A(N)).$$

The two inequalities above used in (30.47) and (30.48) give

$$(30.49)\qquad \begin{aligned}&\left(\int_1^{N-1} + \int_{N+1}^{\infty}\right) A(u)B(u)\,du \\ &\quad = \frac{x}{\alpha(m(\log x))}\left(\int_0^{e^{\phi(N-1)}/x} + \int_{e^{\phi(N+1)}/x}^{\infty}\right)\log\left|1 - \frac{1}{t^2}\right| dt + O(A(N)).\end{aligned}$$

Let

$$(30.50)\qquad g = e^{\phi(N-1)}/x, \qquad h = e^{\phi(N+1)}/x.$$

Then since $|y| < \frac{1}{2}\phi'(N)$,

$$(30.51)\qquad \begin{aligned} g = e^{\phi(N-1)-\phi(N)-y} &= e^{-\phi'(N)-y+O((\phi'(N))^2)} \\ &= 1 - y - \phi'(N) + O((\phi'(N))^2).\end{aligned}$$

Similarly

$$(30.52)\qquad h = e^{\phi(N+1)-\phi(N)-y} = 1 - y + \phi'(N) + O((\phi'(N))^2).$$

Since by (29.11) $\phi'(N) \to 0$ as $N \to \infty$,

$$(30.53)\qquad \tfrac{1}{2} < g < h < \tfrac{3}{2}$$

for large N.

From

$$\int_0^{\infty} \log\left|1 - \frac{1}{t^2}\right| dt = 0,$$

it follows that

$$\begin{aligned}\left(\int_0^{g} + \int_h^{\infty}\right)\log\left|1 - \frac{1}{t^2}\right| dt &= -\int_g^h \log\left|1 - \frac{1}{t^2}\right| dt \\ &= -\int_g^h \log|t-1|\,dt - \int_g^h \log\frac{1+t}{t^2}\,dt.\end{aligned}$$

Using (30.53) this gives

$$(30.54)\qquad \begin{aligned}\left(\int_0^{g} + \int_h^{\infty}\right)\log\left|1 - \frac{1}{t^2}\right| dt &= -\int_{g-1}^{h-1} \log|t|\,dt + O(h-g) \\ &= -(h-1)\log|h-1| \\ &\qquad + (g-1)\log|g-1| + O(h-g).\end{aligned}$$

Using (30.51) and (30.52) this becomes

$$\left(\int_0^g + \int_h^\infty\right) \log\left|1 - \frac{1}{t^2}\right| dt = -(\phi'(N) - y) \log|\phi'(N) - y + O((\phi'(N))^2)|$$
$$- (\phi'(N) + y) \log|\phi'(N) + y + O((\phi'(N))^2)|$$
$$+ O(\phi'(N))$$
$$= -(\phi'(N) - y) \log(\phi'(N) - y)$$
$$- (\phi'(N) + y) \log(\phi'(N) + y) + O(\phi'(N))$$
$$= -\phi'(N) \log\{(\phi'(N))^2 - y^2\}$$
$$+ y \log \frac{\phi'(N) - y}{\phi'(N) + y} + O(\phi'(N))$$
$$= -\phi'(N) \log\{(\phi'(N))^2 - y^2\} + O(\phi'(N)).$$

Using this in (30.49),

$$\left(\int^{N-1} + \int_{N+1}^\infty\right) A(u)B(u)\,du = \frac{x\phi'(N)}{\alpha(m(\log x))} \{-\log[(\phi'(N))^2 - y^2] + O(1)\} + O(A(N)). \tag{30.55}$$

Clearly

$$\phi(N) + y \leqq \phi(N) + \tfrac{1}{2}\phi'(N + 1) < \phi(N + 1),$$
$$\phi(N) + y \geqq \phi(N) - \tfrac{1}{2}\phi'(N) > \phi(N - 1).$$

Since $\log x = \phi(N) + y$,

$$m(\phi(N - 1)) < m(\log x) < m(\phi(N + 1)).$$

Or since m and ϕ are inverse functions,

$$N - 1 < m(\log x) < N + 1.$$

Since α is increasing, this gives

$$\alpha(N - 1) < \alpha(m(\log x)) < \alpha(N + 1).$$

But

$$|\alpha(N \pm 1) - \alpha(N)| \leqq (\phi'(N - 1))^2 \leqq \left(\frac{\phi'(N)}{C_1}\right)^2.$$

From the last two results

$$\alpha(m(\log x)) = \alpha(N) + O((\phi'(N))^2).$$

Thus

$$\frac{1}{\alpha(m(\log x))} = \frac{1}{\alpha(N)} + O\left(\frac{(\phi'(N))^2}{\alpha(N)}\right).$$

Also

$$x = e^{\phi(N)+y} = e^{\phi(N)}(1 + O(\phi'(N))).$$

Using the last two results in (30.55),

$$\left(\int_{1}^{N-1} + \int_{N+1}^{\infty}\right) A(u)B(u)\,du = -A(N)\log\{(\phi'(N))^2 - y^2\} + O(A(N)). \tag{30.56}$$

Thus (30.41) becomes

$$\begin{aligned}\log|G(x)| &\leqq A(N)\log|1 - e^{2v}| + \tfrac{1}{2}A(N-1)\log|1 - x^2e^{2\phi(N-1)}| \\ &\quad + \tfrac{1}{2}A(N+1)\log|1 - x^2e^{-2\phi(N+1)}| \\ &\quad - A(N)\log\{(\phi'(N))^2 - y^2\} + C_{15}A(N).\end{aligned} \tag{30.57}$$

By (30.04), since $A'(u)$ is increasing for large u,

$$A(N-1) = A(N)(1 + O(\phi'(N))),$$

and from this

$$A(N+1) = A(N)(1 + O(\phi'(N))).$$

Using (30.51) and (30.52),

$$\begin{aligned}\log|1 - x^2e^{-2\phi(N\pm1)}| &= \log|2\phi'(N) \mp 2y + O((\phi'(N))^2)| \\ &= \log(\phi'(N) \mp y) + O(1).\end{aligned}$$

Using these results in (30.57)

$$\begin{aligned}\log|G(x)| &\leqq A(N)\{\log|1 - e^{2v}| + \tfrac{1}{2}\log(\phi'(N) + y) \\ &\qquad + \tfrac{1}{2}\log(\phi'(N) - y) - \log((\phi'(N))^2 - y^2) + C_{16}\} \\ &\leqq A(N)\left\{\log\frac{1 - e^{2v}}{\phi'(N)} - \tfrac{1}{2}\log\frac{(\phi'(N))^2 - y^2}{(\phi'(N))^2} + C_{16}\right\} \\ &\leqq A(N)\left\{\log\frac{1 - e^{2v}}{\phi'(N)} + C_{17}\right\}.\end{aligned}$$

Take δ sufficiently small so that if

$$|y| < 2\delta\phi'(N),$$

then $|y| < \frac{1}{2}\phi'(N+1)$, as is possible by (29.12). Then the above two inequalities give

$$\log|G(x)| \leqq A(N)(\log\delta + C_{18}).$$

But this implies that for some $\delta > 0$

$$|G(x)| \leqq 1, \qquad e^{\phi(n) - 2\delta\phi'(n)} \leqq x \leqq e^{\phi(n) + 2\delta\phi'(n)},$$

as $n \to \infty$. This gives (29.14) since $G(x)$ is even.

To complete the proof of the lemma it remains to prove that $G(z)$ is of zero type. Clearly if $r = |z|$,

$$\log |G(z)| \leqq \sum_1^\infty (A(n) + 1) \log\left(1 + \frac{r^2}{e^{2\phi(n)}}\right).$$

Since $A'(u) > 0$ for large u there exists some integer n_1, such that $A(u)$ is increasing for $n > n_1$. Also since $\phi'(u)$ is bounded, there exists a C_{19} such that

$$e^{2\phi(u)} \leqq C_{19} e^{2\phi(n)}, \qquad n \leqq u \leqq n + 1.$$

Thus for larger r,

$$\begin{aligned}\log |G(z)| &\leqq \sum_1^{n_1} (A(n) + 1) \log\left(1 + \frac{r^2}{e^{2\phi(n)}}\right) + 2\int_{n_1}^\infty A(u) \log\left(1 + \frac{C_{19} r^2}{e^{2\phi(u)}}\right) du \\ &\leqq C_{20} \log r + 2\int_1^\infty \frac{e^{\phi(u)}\phi'(u)}{\alpha(u)} \log\left(1 + \frac{C_{19} r^2}{e^{2\phi(u)}}\right) du.\end{aligned}$$

Let $v = e^{\phi(u)}$ and let $\alpha(u) = \beta(v)$. Since v increases with u it follows that, like $\alpha(u)$, $\beta(v)$ is monotone increasing and $\beta(v) \to \infty$ as $v \to \infty$. Thus

$$\begin{aligned}\log |G(z)| &\leqq C_{21} \log r + 2\int_1^\infty \log\left(1 + \frac{C_{19} r^2}{v^2}\right) \frac{dv}{\beta(v)} \\ &\leqq C_{21} \log r + \frac{2}{\beta(1)} \int_1^{r^{1/2}} \log\left(1 + \frac{C_{19} r^2}{v^2}\right) dv \\ &\qquad + \frac{2}{\beta(r^{1/2})} \int_{r^{1/2}}^\infty \log\left(1 + \frac{C_{19} r^2}{v^2}\right) dv \\ &\leqq C_{21} \log r + \frac{2r^{1/2}}{\beta(1)} \log(1 + C_{19} r^2) + \frac{2}{\beta(r^{1/2})} \int_{r^{1/2}}^\infty \log\left(1 + \frac{C_{19} r^2}{v^2}\right) dv \\ &\leqq C_{21} \log r + \frac{2r^{1/2}}{\beta(1)} \log(1 + C_{19} r^2) + \frac{2}{\beta(r^{1/2})} \int_0^\infty \log\left(1 + \frac{C_{19} r^2}{v^2}\right) dv \\ &\leqq C_{22} r^{3/4} + \frac{C_{23} r}{\beta(r^{1/2})}.\end{aligned}$$

Since $\beta(v) \to \infty$ as $v \to \infty$ it follows from the above inequality that

$$\limsup_{|z| \to \infty} \frac{\log |G(z)|}{|z|} \leqq 0.$$

This completes the proof of the lemma.

31. Proof of the main theorem. We prove first a lemma.

LEMMA 31.1. *Let $\{\lambda_n\}$ be the sequence $\{e^{\phi(n)} + k\}$, $(|k| \leqq \delta e^{\phi(n)}\phi'(n)$, $0 < n < \infty)$, where k and n are integers and $\phi(u)$ and δ are defined as in Lemma 29.1. Let*

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right). \tag{31.01}$$

Then

$$(31.02) \qquad \log \frac{1}{|F'(\lambda_n)|} = O(\lambda_n \phi'(m(\log \lambda_n)))$$

where $m(u)$ is the inverse function of $\phi(u)$.

Proof of Lemma 31.1. From the definition of $\{\lambda_n\}$, to every λ_m there corresponds an N such that

$$|\lambda_m - e^{\phi(N)}| \leqq \delta e^{\phi(N)}\phi'(N).$$

Thus from (31.01)

$$(31.03) \qquad \begin{aligned} \log|F'(\lambda_m)| &= \left(\sum_{n=1}^{N-1} + \sum_{N+1}^{\infty}\right) \sum_{|k| \leqq \delta e^{\phi(n)}\phi'(n)} \log\left|1 - \frac{\lambda_m^2}{(e^{\phi(n)} + k)^2}\right| \\ &\quad + \sum_{|k| \leqq \delta e^{\phi(N)}\phi'(N)}{}' \log\left|1 - \frac{\lambda_m^2}{(e^{\phi(N)} + k)^2}\right| + \log\frac{2}{\lambda_m} \end{aligned}$$

where the prime on the last summation sign denotes omission of the term $e^{\phi(N)} + k = \lambda_m$.

Clearly for $n < N$,

$$(31.04) \qquad \begin{aligned} I_n(k) &= \log\left(\frac{\lambda_m^2}{(e^{\phi(n)} + k)^2} - 1\right)\left(\frac{\lambda_m^2}{(e^{\phi(n)} - k)^2} - 1\right) - \log\left(\frac{\lambda_m^2}{e^{2\phi(n)}} - 1\right)^2 \\ &= \log\frac{\lambda_m^4 - 2\lambda_m^2(e^{2\phi(n)} + k^2) + (e^{2\phi(n)} - k^2)^2}{(\lambda_m^2 - e^{2\phi(n)})^2} \frac{e^{4\phi(n)}}{(e^{2\phi(n)} - k^2)^2} \\ &\geqq \log\frac{\lambda_m^4 - 2\lambda_m^2(e^{2\phi(n)} + k^2) + (e^{2\phi(n)} - k^2)^2}{(\lambda_m^2 - e^{2\phi(n)})^2} \\ &= \log\left(1 - \frac{2k^2(e^{2\phi(n)} + \lambda_m^2) - k^4}{(\lambda_m - e^{\phi(n)})^2(\lambda_m + e^{\phi(n)})^2}\right). \end{aligned}$$

Clearly

$$\lambda_m \geqq e^{\phi(N)}(1 - \delta\phi'(N)).$$

For small $x > 0$, $1 - x > e^{-2x}$. Therefore for large N,

$$\lambda_m > e^{\phi(N) - 2\delta\phi'(N)}.$$

Since $\phi'(u)$ is decreasing,

$$\phi(N) - 2\delta\phi'(N) > \phi(N - 2\delta)$$

and therefore

$$\lambda_m > e^{\phi(N-2\delta)}.$$

Thus for large n

$$(31.05) \qquad \begin{aligned} \lambda_m - e^{\phi(n)} > e^{\phi(N-2\delta)} - e^{\phi(n)} &= \int_n^{N-2\delta} e^{\phi(u)}\phi'(u)\,du \\ &\geqq (N - 2\delta - n)e^{\phi(n)}\phi'(n), \end{aligned}$$

since $e^{\phi(u)}\phi'(u)$ is an increasing function for large u. Using this (31.04) becomes

$$\begin{aligned}
I_n(k) &> \log\left(1 - \frac{2(\lambda_m^2 + e^{2\phi(n)})k^2 - k^4}{(N - 2\delta - n)^2 e^{2\phi(n)}(\phi'(n))^2(\lambda_m + e^{\phi(n)})^2}\right) \\
&> \log\left(1 - \frac{2(\lambda_m^2 + e^{2\phi(n)})k^2}{(N - 2\delta - n)^2 e^{2\phi(n)}(\phi'(n))^2(\lambda_m + e^{\phi(n)})^2}\right) \\
&> \log\left(1 - \frac{2k^2}{(N - 2\delta - n)^2 e^{2\phi(n)}(\phi'(n))^2}\right) \\
&\geqq \log\left(1 - \frac{2\delta^2}{(N - 2\delta - n)^2}\right).
\end{aligned}$$

Since $N > n$ and δ is small, it follows that

$$I_n(k) \geqq \frac{-4\delta^2}{(N - n - 2\delta)^2}.$$

Thus for large n,

$$\sum_{1 \leqq k \leqq \delta e^{\phi(n)}\phi'(n)} I_n(k) \geqq \frac{-4\delta^3\phi'(n)e^{\phi(n)}}{(N - n - 2\delta)^2}.$$

From this and the definition of $I_n(k)$ in (31.04), using $[x]$ to represent the largest integer not exceeding x,

$$\begin{aligned}
J_1 &= \sum_{n=1}^{N-1} \sum_{|k| \leq \delta e^{\phi(n)}\phi'(n)} \log\left(\frac{\lambda_m^2}{(e^{\phi(n)} + k)^2} - 1\right) \\
&\geqq \sum_{n=1}^{N-1} ([2\delta\phi'(n)e^{\phi(n)}] + 1) \log\left(\frac{\lambda_m^2}{e^{2\phi(n)}} - 1\right) \\
&\qquad - 4\delta^3\phi'(N)e^{\phi(N)} \sum_1^{N-1} \frac{1}{(N - n - 2\delta)^2} - C_{23},
\end{aligned}$$

where C_{23} compensates for those $I_n(k)$ where n is small. It follows easily that

$$\begin{aligned}
J_1 &\geqq \sum_{n=1}^{N-1} 2\delta\phi'(n)e^{\phi(n)} \log\left(\frac{\lambda_m^2}{e^{2\phi(n)}} - 1\right) \\
&\qquad - 3\sum_{n=1}^{N-1} \left|\log\left(\frac{\lambda_m^2}{e^{2\phi(n)}} - 1\right)\right| - 10\phi'(N)e^{\phi(N)} - C_{23}.
\end{aligned} \tag{31.06}$$

Since

$$\log\left(\frac{\lambda_m^2}{e^{2\phi(n)}} - 1\right), \qquad 1 \leqq n \leqq N - 1,$$

takes its extreme values for $n = 1$ and $n = N - 1$,

$$\max_{1 \leqq n < N} \left|\log\left(\frac{\lambda_m^2}{e^{2\phi(n)}} - 1\right)\right| \leqq 2\log\frac{\lambda_m}{e^{\phi(1)}} + \log\frac{e^{\phi(N-1)}}{\lambda_m - e^{\phi(N-1)}}.$$

Using (31.05) this becomes

$$\max_{1\leq n<N}\left|\log\left(\frac{\lambda_m^2}{e^{2\phi(n)}}-1\right)\right| = O\left(\log\lambda_m + \log\frac{1}{\phi'(N)}\right).$$

But $\lambda_m < 2e^{\phi(N)}$ for large m, and by (29.10), $1/\phi'(N) < N$, for large N. Thus

$$\max_{1\leq n<N}\left|\log\left(\frac{\lambda_m^2}{e^{2\phi(n)}}-1\right)\right| = O(\phi(N)+\log N) = O(\phi(N)),$$

since for large N, $\log N < \phi(N)$ by (29.09). Thus (31.06) becomes

$$J_1 > \sum_{n=1}^{N-1} 2\delta\phi'(n)e^{\phi(n)}\log\left(\frac{\lambda_m^2}{e^{2\phi(n)}}-1\right) - C_{24}N\phi(N) - 10\phi'(N)e^{\phi(N)} - C_{25}.$$

By (29.09) and (29.10), $N\phi(N) = O(\phi'(N)e^{\phi(N)})$. Thus for large N,

$$J_1 \geqq \sum_{n=1}^{N-1} 2\delta\phi'(n)e^{\phi(n)}\log\left(\frac{\lambda_m^2}{e^{2\phi(n)}}-1\right) - C_{26}\phi'(N)e^{\phi(N)}. \tag{31.07}$$

For $n > N$, it is easily shown that

$$\begin{aligned}
I_n(k) &= \log\left(1-\frac{\lambda_m^2}{(e^{\phi(n)}+k)^2}\right)\left(1-\frac{\lambda_m^2}{(e^{\phi(n)}-k)^2}\right) - \log\left(1-\frac{\lambda_m^2}{e^{2\phi(n)}}\right)^2 \\
&= \log\left\{1+\frac{2k^2\lambda_m^4 - 6e^{2\phi(n)}\lambda_m^2k^2}{(e^{2\phi(n)}-k^2)(e^{2\phi(n)}-\lambda_m^2)^2} + \frac{k^4(\lambda_m^4 - 4e^{2\phi(n)}\lambda_m^2)}{(e^{2\phi(n)}-\lambda_m^2)^2(e^{2\phi(n)}-k^2)^2}\right\} \\
&\geqq \log\left\{1-\frac{6k^2\lambda_m^2e^{2\phi(n)}}{(e^{2\phi(n)}-k^2)(e^{\phi(n)}+\lambda_m)^2(e^{\phi(n)}-\lambda_m)^2}\right. \\
&\qquad\left. -\frac{4k^4\lambda_m^2e^{2\phi(n)}}{(e^{\phi(n)}-\lambda_m)^2(e^{\phi(n)}+\lambda_m)^2(e^{2\phi(n)}-k^2)^2}\right\} \\
&\geqq \log\left\{1-\frac{6k^2\lambda_m^2}{(e^{2\phi(n)}-k^2)(e^{\phi(n)}-\lambda_m)^2} - \frac{4k^4\lambda_m^2}{(e^{\phi(n)}-\lambda_m)^2(e^{2\phi(n)}-k^2)^2}\right\}.
\end{aligned}$$

Since $k = o(e^{\phi(n)})$ for large n

$$\begin{aligned}
I_n(k) &\geqq \log\left\{1-\frac{10k^2\lambda_m^2}{e^{2\phi(n)}(e^{\phi(n)}-\lambda_m)^2} - \frac{6k^4\lambda_m^2}{e^{4\phi(n)}(e^{\phi(n)}-\lambda_m)^2}\right\} \\
&\geqq \log\left\{1-\frac{16k^2\lambda_m^2}{e^{2\phi(n)}(e^{\phi(n)}-\lambda_m)^2}\right\}.
\end{aligned}$$

Since

$$e^{\phi(n)} - \lambda_m > (n-N+2\delta)\phi'(N)e^{\phi(N)} > 0 \tag{31.08}$$

in much the same way as in (31.05), and since $\log(1-x) > -2x$ for small $x > 0$,

$$I_n(k) \geqq \frac{-32k^2\lambda_m^2}{e^{2\phi(n)}(e^{\phi(n)}-\lambda_m)^2}.$$

Therefore

$$
\begin{aligned}
\sum_{1\le k\le \delta\phi'(n)e^{\phi(n)}} I_n(k) &\ge \frac{-32\lambda_m^2}{e^{2\phi(n)}(e^{\phi(n)}-\lambda_m)^2}\sum_{1\le k\le\delta\phi'(n)e^{\phi(n)}} k^2\\
&\ge \frac{-32\lambda_m^2}{e^{2\phi(n)}(e^{\phi(n)}-\lambda_m)^2}\,\delta^3(\phi'(n))^3 e^{3\phi(n)}\\
&= -32\delta^3(\phi'(n))^2\lambda_m \frac{\lambda_m\phi'(n)e^{-\phi(n)}}{(1-\lambda_m e^{-\phi(n)})^2}\\
&\ge -32\delta^3(\phi'(N))^2\lambda_m\frac{\lambda_m\phi'(n)e^{-\phi(n)}}{(1-\lambda_m e^{-\phi(n)})^2}.
\end{aligned}
\tag{31.09}
$$

Since $\phi'(u)e^{-\phi(u)}$ is decreasing for large u

$$
\begin{aligned}
&(\phi'(N))^2\lambda_m\sum_{N+1}^{\infty} \frac{\phi'(n)e^{-\phi(n)}}{(1-\lambda_m e^{-\phi(n)})^2}\\
&\qquad\le (\phi'(N))^2\lambda_m\frac{\phi'(N+1)e^{-\phi(N+1)}}{(1-\lambda_m e^{-\phi(N+1)})^2} + (\phi'(N))^2\int_{N+1}^{\infty} \frac{\phi'(u)e^{-\phi(u)}\lambda_m\,du}{(1-\lambda_m e^{-\phi(u)})^2}\\
&\qquad\le \frac{\lambda_m(\phi'(N))^2 e^{\phi(N+1)}\phi'(N+1)}{(e^{\phi(N+1)}-\lambda_m)^2} + \frac{(\phi'(N))^2}{1-\lambda_m e^{-\phi(N+1)}}.
\end{aligned}
$$

By (31.08)

$$e^{\phi(N+1)}-\lambda_m > \tfrac{1}{2}\phi'(N)e^{\phi(N)}$$

Thus (for large N),

$$
\begin{aligned}
&(\phi'(N))^2\lambda_m\sum_{N+1}^{\infty} \frac{\phi'(n)e^{-\phi(n)}}{(1-\lambda_m e^{-\phi(n)})^2}\\
&\qquad\le 4\phi'(N+1)\frac{\lambda_m}{e^{\phi(N+1)}}e^{2\phi(N+1)-2\phi(N)} + 2\phi'(N)e^{\phi(N+1)-\phi(N)}\\
&\qquad\le 4\phi'(N+1)e^{2\phi'(N)} + 2\phi'(N)e^{\phi'(N)} < 10\phi'(N)
\end{aligned}
$$

Combining this result with (31.09)

$$
\begin{aligned}
\sum_{n=N+1}^{\infty}\sum_{1\le k\le\phi'(n)e^{\phi(n)}} I_n(k) &\ge -320\delta^3\lambda_m\phi'(N)\\
&> -10\phi'(N)e^{\phi(N+1)} > -20\phi'(N)e^{\phi(N)}
\end{aligned}
$$

Recalling the definition of $I_n(k)$, this implies

$$
\begin{aligned}
&\sum_{n=N+1}^{\infty}\sum_{|k|\le\delta\phi'(n)e^{\phi(n)}}\log\left(1-\frac{\lambda_m^2}{(e^{\phi(n)}+k)^2}\right)\\
&\qquad\ge \sum_{n=N+1}^{\infty}(2\delta\phi'(n)e^{\phi(n)}+1)\log\left(1-\frac{\lambda_m^2}{e^{2\phi(n)}}\right) - 20\phi'(N)e^{\phi(N)}.
\end{aligned}
\tag{31.10}
$$

Combining (31.03), (31.07) and (31.10)

$$\log |F'(\lambda_m)| \geqq 2\delta\left(\sum_1^{N-1} + \sum_{N+1}^{\infty}\right)\phi'(n)e^{\phi(n)} \log\left(1 - \frac{\lambda_m^2}{e^{2\phi(n)}}\right)$$

(31.11)
$$+ \sum_{N+1}^{\infty} \log\left(1 - \frac{\lambda_m^2}{e^{2\phi(n)}}\right) + \sum_{|k| \leqq \delta\phi'(N)e^{\phi(N)}}{}' \log\left|1 - \frac{\lambda_m^2}{(e^{\phi(N)} + k)^2}\right| - C_{27}\phi'(N)\,e^{\phi(N)}.$$

Clearly

$$\sum_{N+1}^{\infty} \log\left(1 - \frac{\lambda_m^2}{e^{2\phi(n)}}\right) \geqq N \log\left(1 - \frac{\lambda_m^2}{e^{2\phi(N+1)}}\right) + \sum_{2N+1}^{\infty} \log\left(1 - \frac{\lambda_m^2}{e^{2\phi(n)}}\right)$$

$$\geqq N \log\left(\frac{e^{\phi(N+1)} - \lambda_m}{e^{\phi(N+1)}}\right) + \sum_{2N+1}^{\infty} \log\left(1 - \frac{e^{2\phi(N+1)}}{e^{2\phi(n)}}\right)$$

$$\geqq N \log\left(\frac{e^{\phi(N+1)} - \lambda_m}{e^{\phi(N+1)}}\right) + \sum_{2N+1}^{\infty} \log\left(1 - e^{-2(n-N-1)\phi'(n)}\right)$$

$$\geqq N \log\left(\frac{e^{\phi(N+1)} - \lambda_m}{e^{\phi(N+1)}}\right) + \sum_{2N}^{\infty} \log\left(1 - e^{-n\phi'(n)}\right).$$

By (31.08), (29.10), and later (29.09)

$$\sum_{N+1}^{\infty} \log\left(1 - \frac{\lambda_m^2}{e^{2\phi(n)}}\right) \geqq N \log \frac{\frac{1}{2}\phi'(N)\,e^{\phi(N)}}{e^{\phi(N+1)}} + \sum_{2N}^{\infty} \log\left(1 - e^{\epsilon\phi(n)}\right)$$

$$\geqq N \log \tfrac{1}{4}\phi'(N) - \sum_{2N}^{\infty} 2e^{-\epsilon\phi(n)}$$

$$\geqq -N \log \frac{4}{\phi'(N)} - 2\sum_{2N}^{\infty} e^{-\epsilon n^a}.$$

Since by (29.10), $4/\phi'(u) < u$ for large u,

(31.12)
$$\sum_{N+1}^{\infty} \log\left(1 - \frac{\lambda_m^2}{e^{2\phi(n)}}\right) \geqq -N \log N - C_{28} \geqq -C_{29}\phi'(N)\,e^{\phi(N)},$$

where this last inequality also involves the use of (29.09), $u^a < \phi(u)$.

For large N, using $\sum'$ without the range of summation indicated to cover the same range as $\sum'$ in (31.11), we have

$$\sum{}' \log\left|1 - \frac{\lambda_m^2}{(e^{\phi(N)} + k)^2}\right| \geqq \sum{}' \log\left|1 - \frac{\lambda_m}{e^{\phi(N)} + k}\right|$$

$$\geqq \sum{}' \log|e^{\phi(N)} + k - \lambda_m| - \sum{}' \log(2e^{\phi(N)})$$

$$\geqq \sum{}' \log|e^{\phi(N)} + k - \lambda_m| - 2\delta\phi'(N)e^{\phi(N)} \log(2e^{\phi(N)}).$$

But as k varies over its range $e^{\phi(N)} + k - \lambda_m$ takes on integer values from $-n_1$ to n_2 where

(31.13)
$$n_1 + n_2 = [2\delta\phi'(N)e^{\phi(N)}].$$

Thus

$$\sum{}' \log \left| 1 - \frac{\lambda_m^2}{(e^{\phi(N)} + k)^2} \right| \geqq \left(\sum_{j=1}^{n_1} + \sum_{j=1}^{n_2} \right) \log j - 2\delta\phi'(N)e^{\phi(N)}(\phi(N) + 1)$$

$$\geqq \left(\int_1^{n_1} + \int_1^{n_2} \right) \log u \, du - 2\delta\phi'(N)\phi(N)e^{\phi(N)} - 2\delta\phi'(N)e^{\phi(N)}$$

$$\geqq n_1 \log n_1 + n_2 \log n_2 - n_1 - n_2 - 2\delta\phi'(N)\phi(N)e^{\phi(N)} - 2\delta\phi'(N)e^{\phi(N)}$$

$$\geqq n_1 \log n_1 + n_2 \log n_2 - 2\delta\phi'(N)\phi(N)e^{\phi(N)} - 4\delta\phi'(N)e^{\phi(N)}.$$

For fixed N, $n_1 + n_2$ is fixed. By differentiating

$$x \log x + (c - x) \log (c - x), \qquad 1 \leqq x \leqq c - 1,$$

it follows easily that it takes its least value for $x = 1$. Thus using (31.13),

$$n_1 \log n_1 + n_2 \log n_2 \geqq (2\delta\phi'(N)e^{\phi(N)} - 2) \log (2\delta\phi'(N)e^{\phi(N)} - 2).$$

Therefore for large N

$$\sum{}' \log \left| 1 - \frac{\lambda_m^2}{(e^{\phi(N)} + k)^2} \right| \geqq 2\delta\phi'(N)e^{\phi(N)} \log (2\delta\phi'(N)e^{\phi(N)} - 2)$$

$$- 2\delta\phi'(N)\phi(N)e^{\phi(N)} - 6\phi'(N)e^{\phi(N)}$$

$$\geqq 2\delta\phi'(N)e^{\phi(N)} \{\log \phi'(N)e^{\phi(N)} + \log (2\delta - 2e^{-\phi(N)}/\phi'(N))\}$$

$$- 2\delta\phi'(N)\phi(N)e^{\phi(N)} - 6\phi'(N)e^{\phi(N)}$$

$$\geqq 2\delta\phi'(N)e^{\phi(N)} \log \phi'(N) - 10\phi'(N)e^{\phi(N)}.$$

Using this and (31.12), (31.11) becomes

$$\log |F(\lambda_m)| \geqq 2\delta \left(\sum_1^{N-1} + \sum_{N+1}^{\infty} \right) \phi'(n)e^{\phi(n)} \log \left| 1 - \frac{\lambda_m^2}{e^{2\phi(n)}} \right| \tag{31.14}$$

$$+ 2\delta\phi'(N)e^{\phi(N)} \log \phi'(N) - C_{30}\phi'(N)e^{\phi(N)}.$$

By the same proof as used in obtaining Lemma 30.2, in somewhat simpler form since here $\alpha(u) = 1$, it follows that

$$\left(\sum_1^{N-1} + \sum_{N+1}^{\infty} \right) \phi'(n)e^{\phi(n)} \log \left| 1 - \frac{\lambda_m^2}{e^{2\phi(n)}} \right| = \tfrac{1}{2}\phi'(N-1)e^{\phi(N-1)} \log \left| 1 - \frac{\lambda_m^2}{e^{2\phi(N-1)}} \right|$$

$$+ \tfrac{1}{2}\phi'(N+1)e^{\phi(N+1)} \log \left| 1 - \frac{\lambda_m^2}{e^{2\phi(N+1)}} \right| \tag{31.15}$$

$$+ \left(\int_1^{N-1} + \int_{N+1}^{\infty} \right) \phi'(u)e^{\phi(u)} \log \left| 1 - \frac{\lambda_m^2}{e^{2\phi(u)}} \right| du + O(\phi'(N)e^{\phi(N)}).$$

If the right side of (31.15) is handled by the same method used in the proof of Lemma 29.1 with the very great simplification that here $\alpha(u)$ is not involved, (31.15) becomes

$$\left(\sum_{1}^{N-1} + \sum_{N+1}^{\infty}\right) \phi'(n)e^{\phi(n)} \log\left|1 - \frac{\lambda_m^2}{e^{2\phi(n)}}\right| = -\phi'(N)e^{\phi(N)} \log \phi'(N) + O(\phi'(N)e^{\phi(N)}).$$

Thus (31.14) becomes for large m,

$$\log | F'(\lambda_m) | \geqq -C_{31}\phi'(N)e^{\phi(N)} \geqq -2C_{31}\phi'(N)\lambda_m \tag{31.16}$$

since

$$| \lambda_m - e^{\phi(N)} | \leqq \delta\phi'(N)e^{\phi(N)}.$$

The last result can be written as

$$| e^{\log \lambda_m - \phi(N)} - 1 | \leqq \delta\phi'(N).$$

Thus as $m \to \infty$

$$\log \lambda_m - \phi(N) \to 0.$$

Therefore for large m

$$\phi(N) > \tfrac{1}{2} \log \lambda_m .$$

Or

$$m(\phi(N)) > m[\tfrac{1}{2} \log \lambda_m],$$

which, since $m(u)$ is the inverse function of $\phi(u)$, can be written as

$$N > m[\tfrac{1}{2} \log \lambda_m].$$

Since $\phi'(u)$ is decreasing, it follows from (31.16) that

$$\log | F'(\lambda_m) | > -2C_{31}\phi'\{m(\tfrac{1}{2} \log \lambda_m)\}\lambda_m .$$

Using (30.08),

$$\log | F'(\lambda_m) | > -2C_{31}\phi' \{C_2\, m(\log \lambda_m)\} \lambda_m .$$

Iterating (30.07),

$$\log | F'(\lambda_m) | > -C_{32}\phi'\{m(\log \lambda_m)\}\lambda_m ,$$

which completes the proof of the lemma.

LEMMA 31.2. *If $M(x)$ satisfies the requirements of Theorem XLVII, then there exists a function $\phi(u)$ satisfying the requirements of Lemma 29.1 and for small x*

$$\sum_{\lambda_n > 10}^{\infty} \exp\,[\lambda_n\phi'\{m(\log \lambda_n)\} - \lambda_n x] < M(x), \qquad x > 0, \tag{31.17}$$

where m is the inverse function of ϕ.

Proof of Lemma 31.2. We shall deal mainly with $\mu(x) = \log \log M(x)$. Let

$$\mu_1(x) = \mu(x), \quad \left(\mu(x) \leqq \frac{1}{x}\right), \qquad \mu_1(x) = \frac{1}{x}, \quad \left(\mu(x) > \frac{1}{x}\right).$$

Let the set of intervals in $(0, 1)$ where $\mu(x) > 1/x$ be denoted by E and let the complementary part of $(0, 1)$ be denoted by $C(E)$. If

$$\int_0^1 \mu_1(x)\,dx < \infty,$$

then

$$\int_E \frac{dx}{x} + \int_{C(E)} \mu(x)\,dx < \infty. \tag{31.18}$$

Since

$$\int_0^1 \mu(x)\,dx = \infty,$$

it follows from (31.18) that

$$\int_E \mu(x)\,dx = \infty. \tag{31.19}$$

Also from (31.18)

$$\int_E \frac{dx}{x} < \infty. \tag{31.20}$$

Since E is composed of intervals, these can be arranged in order of decreasing length and denoted by (a_n, b_n), $(n > 0)$. Then (31.19) gives

$$\sum_{n=1}^{\infty} \int_{a_n}^{b_n} \mu(x)\,dx = \infty. \tag{31.21}$$

But $\mu(a_n) \leqq 1/a_n$ otherwise a_n could not be the end-point of an interval of E. Since $\mu(x)$ is decreasing, it follows that

$$\sum_1^{\infty} \int_{a_n}^{b_n} \mu(x)\,dx \leqq \sum_1^{\infty} \int_{a_n}^{b_n} \frac{1}{a_n}\,dx = \sum_1^{\infty} \left(\frac{b_n}{a_n} - 1\right).$$

By (31.21) this becomes

$$\sum_1^{\infty} \left(\frac{b_n}{a_n} - 1\right) = \infty. \tag{31.22}$$

On the other hand (31.20) gives

$$\sum \log \frac{b_n}{a_n} \leqq \sum \int_{a_n}^{b_n} \frac{dx}{x} < \infty. \tag{31.23}$$

But (31.22) and (31.23) are contradictory. Thus

$$\int_0^1 \mu_1(x)\,dx = \infty.$$

Again since $\mu_1(x)$ is never larger then $\mu(x)$, it follows that if x is in $C(E)$, $\frac{1}{2}\mu(x) \geqq \mu(px)$ implies $\frac{1}{2}\mu_1(x) > \mu_1(px)$. If x is in E, then $\mu_1(x) = 1/x$ and $\mu_1(px) \leqq 1/(px)$. Since $p > 2$, $\frac{1}{2}\mu_1(x) > \mu_1(px)$. Thus

$$\mu_1(x) \leqq \mu(x), \qquad \mu_1(x) \leqq 1/x,$$

and in addition satisfies the requirements of $\mu(x)$.

Let

$$\mu_2(x) = x \int_x^\infty \frac{\mu_1(y)}{y^2}\, dy.$$

Then $\mu_2(x) < \mu_1(x)$ and is differentiable. Differentiating, it follows easily that $\mu_2(x)$ is decreasing. Using $\mu_1(y) < \frac{1}{2}\mu_1(y/p)$,

$$\mu_2(px) = px \int_{px}^\infty \frac{\mu_1(y)}{y^2}\, dy < \frac{px}{2} \int_{px}^\infty \frac{\mu_1(y/p)}{y^2}\, dy$$

$$= \tfrac{1}{2}x \int^\infty \frac{\mu_1(t)}{t^2}\, dt = \tfrac{1}{2}\mu_2(x).$$

That is,

$$\mu_2(px) < \tfrac{1}{2}\mu_2(x).$$

Also,

$$\int_0^1 \mu_2(x)\, dx \geqq \int_0^1 dx x \int_x^1 \frac{\mu_1(y)}{y^2}\, dy = \frac{1}{2} \int_0^1 \mu_1(y)\, dy = \infty.$$

Let

$$\mu_3(x) = x \int_x^\infty \frac{\mu_2(y)}{y^2}\, dy.$$

Then $\mu_3(x)$ bears the same relation to $\mu_2(x)$ as $\mu_2(x)$ does to $\mu_1(x)$. Thus $\mu_3(x)$ is decreasing and is twice differentiable. Also

(31.24) $$\mu_3(x) < 1/x, \qquad \mu_3(x) < \mu(x),$$

$$\mu_3(px) < \tfrac{1}{2}\mu_3(x), \qquad \int_0^1 \mu_3(x)\, dx = \infty.$$

Clearly

$$\int_x^1 \mu_3(y)\, dy \leqq \sum_{k=1}^K \int_{xp^{k-1}}^{xp^k} \mu_3(y)\, dy$$

where K is so chosen that

$$1 \leqq xp^K < p.$$

Thus

$$\int_x^1 \mu_3(y)\,dy \leqq \sum_{k=1}^{K} (xp^k - xp^{k-1})\mu_3(xp^{k-1}) = x(p-1)\sum_{k=1}^{K} \mu_3(xp^{k-1})p^{k-1}.$$

By (31.24)

$$\mu_3(x) > 2\mu_3(px) > 4\mu_3(p^2x) > \cdots.$$

Thus

$$\int_x^1 \mu_3(y)\,dy < x(p-1)\mu_3(x)\sum_{k=1}^{K}\left(\frac{p}{2}\right)^{k-1} < x\mu_3(x)(p-1)\frac{(p/2)^K}{\frac{1}{2}p - 1}.$$

But $p^K < p/x$ and $p > 4$. Thus

$$\int_x^1 \mu_3(y)\,dy < (p-1)\mu_3(x)2^{-K}p.$$

But $p^K \geqq 1/x$ and therefore

$$2^K = p^{K\log 2/\log p} \geqq \left(\frac{1}{x}\right)^{\log 2/\log p}.$$

Thus

$$\int_x^1 \mu_3(y)\,dy \leqq p(p-1)x^{-\log 2/\log p}\mu_3(x).$$

Letting $x \to 0$ and setting $b = \log 2/\log p$,

(31.25) $$\lim_{x\to 0} x^{-b}\mu_3(x) = \infty, \qquad b > 0.$$

Let

$$\mu_4(x) = \tfrac{1}{2}\mu_3(x) + \tfrac{1}{2}x^{-b}.$$

Then by (31.24), for $x < 1$,

(31.26) $$\mu_4(x) < \frac{1}{x}, \qquad \mu_4(px) \leqq \tfrac{1}{2}\mu_4(x), \qquad \int_0^1 \mu_4(x)\,dx = \infty,$$

and by (31.24) and (31.25) for small x

$$\mu_4(x) < \mu(x).$$

Also

(31.27) $$-\mu_4'(x) > \tfrac{1}{2}bx^{-b-1}.$$

Let

$$\Phi(x) = -\int_{2x}^{\infty} \frac{\mu_4'(y)}{y}\,dy.$$

Let $\phi'(x)$ be the inverse function of $\Phi(x)$. Let

$$\phi(u) = \int_0^u \phi'(y)\,dy.$$

Then setting $\phi'(u) = x$ and $\phi'(y) = z$, $y = \Phi(z)$,

$$\phi(u)\phi'(u) = \phi'(u) \int_0^u \phi'(y)\,dy = -x \int_x^{\phi'(0)} z\Phi'(z)\,dz$$

$$= -x \int_x^{\phi'(0)} \mu_4'(2z)\,dz \leqq \frac{x}{2}\mu_4(2x).$$

Since $\mu_4(x) < 1/x$ it follows that

$$\phi(u)\phi'(u) = O(1), \qquad u \to \infty.$$

Next consider $\phi''(u)$. Using $\phi'(u) = x$, $u = \Phi(x)$,

$$\phi''(u) = 1/\Phi'(x).$$

Thus

$$\limsup_{u\to\infty} \left| \frac{\phi''(u)}{(\phi'(u))^2} \right| = \limsup_{x\to 0} \frac{-1}{x^2\Phi'(x)}. \tag{31.28}$$

But by (31.27)

$$-\Phi'(x) = -\frac{\mu_4'(2x)}{x} > \tfrac{1}{2}bx^{-b-2}. \tag{31.29}$$

Thus (31.28) gives

$$\phi''(u) = o((\phi'(u))^2), \qquad u \to \infty.$$

Also

$$\phi'''(u) = -\frac{\Phi''(x)}{(\Phi'(x))^2}\frac{dx}{du} = -\frac{\Phi''(x)}{(\Phi'(x))^3}. \tag{31.30}$$

But

$$\Phi''(x) = \frac{d}{dx}\left\{\frac{\mu_4'(2x)}{x}\right\} = \frac{d}{dx}\left(\frac{1}{x}\{-b(2x)^{-b-1} + \mu_3'(2x)\}\right) > \frac{d}{dx}\left(\frac{\mu_3'(2x)}{x}\right).$$

Since μ_3' is negative,

$$\Phi''(x) \geqq \frac{1}{x}\frac{d}{dx}\mu_3'(2x) = \frac{1}{x}\frac{d}{dx}\left\{\int_{2x}^\infty \frac{\mu_2(y)}{y^2}\,dy - \frac{\mu_2(2x)}{2x}\right\}$$

$$= \frac{1}{x}\frac{d}{dx}\int_{2x}^\infty \frac{\mu_2'(y)}{y}\,dy = -\frac{\mu_2'(2x)}{x^2} > 0.$$

That is, $\Phi''(x) > 0$. But $\Phi'(x) < 0$. Thus (31.30) gives

$$\phi'''(u) > 0.$$

We next consider

$$\begin{aligned}
u\phi'(u) - \frac{1}{4p}\phi(u) &= u\phi'(u) - \frac{1}{4p}\int_0^u \phi'(y)\,dy \\
&= x\Phi(x) + \frac{1}{4p}\int_x^{\phi'(0)} z\Phi'(z)\,dz \\
&= x\int_{2x}^{\infty} \frac{\mu_4'(y)}{y}\,dy + \frac{1}{4p}\int_x^{\phi'(0)} \mu_4'(2z)\,dz \\
&\geqq -x\left\{-\frac{\mu_4(2x)}{2x} + \int_{2x}^{\infty}\frac{\mu_4(y)}{y^2}\,dy\right\} - \frac{1}{4p}\mu_4(2x) \\
&= \frac{1}{2}\left(1 - \frac{1}{2p}\right)\mu_4(2x) - x\int_{2x}^{2px}\frac{\mu_4(y)}{y^2}\,dy - x\int_{2px}^{\infty}\frac{\mu_4(y)}{y^2}\,dy \\
&\geqq \frac{1}{2}\left(1 - \frac{1}{2p}\right)\mu_4(2x) - \mu_4(2x)x\int_{2x}^{2px}\frac{dy}{y^2} - x\int_{2px}^{\infty}\frac{\mu_4(y)}{y^2}\,dy \\
&= \frac{1}{4p}\mu_4(2x) - x\int_{2px}^{\infty}\frac{\mu_4(y)}{y^2}\,dy.
\end{aligned}$$

By (31.26), $\mu_4(y) < \frac{1}{2}\mu_4(y/p)$. Thus the above inequality becomes

$$\begin{aligned}
u\phi'(u) - \frac{1}{2p}\phi(u) &\geqq \frac{1}{4p}\mu_4(2x) - \frac{x}{2}\int_{2px}^{\infty}\mu_4(y/p)\frac{dy}{y^2} \\
&\geqq \frac{1}{4p}\mu_4(2x) - \frac{x}{2}\mu_4(2x)\int_{2px}^{\infty}\frac{dy}{y^2} = 0
\end{aligned}$$

Thus for large u,

$$\frac{u\phi'(u)}{\phi(u)} \geqq \frac{1}{2p}.$$

Since $\phi'(u)$ is positive and decreasing,

$$\phi(u) = \int_0^u \phi'(y)\,dy \geqq u\phi'(u).$$

Thus

$$\begin{aligned}
\liminf_{u\to\infty} u^{-a}\phi(u) &\geqq \liminf_{u\to\infty} u^{1-a}\phi'(u) = \liminf_{x\to 0} x(\Phi(x))^{1-a} \\
&= \liminf_{x\to 0}\left\{x\left(\int_{2x}^{\infty}\frac{-\mu_4'(y)}{y}\,dy\right)^{1-a}\right\}.
\end{aligned}$$

Using (31.27) this becomes

$$\begin{aligned}
\liminf_{u\to\infty} u^{-a}\phi(u) &\geqq \liminf_{x\to 0}\left\{x\left(\frac{b}{b+1}\frac{1}{(2x)^{b+1}}\right)^{1-a}\right\} \\
&= \left(\frac{b}{b+1}\right)^{1-a}\left(\frac{1}{2}\right)^{(1+b)(1-a)}\liminf_{x\to 0} x^{a\ \ b+ab}.
\end{aligned}$$

For sufficiently small a it follows that

$$\liminf_{u\to\infty} u^{-a}\phi(u) = \infty.$$

Finally setting $\phi'(y) = z$,

$$\int_1^\infty (\phi'(y))^2\,dy = -\int_0^{\phi'(1)} z^2\Phi'(z)\,dz$$

$$= -\int_0^{\phi'(1)} 3\mu_4'(2z)\,dz$$

$$= -\tfrac{1}{2}z\mu_4(2z)\Big]_0^{\phi'(1)} + \frac{1}{2}\int_0^{\phi'(1)} \mu_4(2z)\,dz.$$

Thus by (31.26) it follows that

$$\int_1^\infty (\phi'(y))^2\,dy = \infty.$$

Thus $\phi(u)$ as defined here satisfies all the requirements of Lemma 29.1.

We now turn to (31.17). Clearly if $\phi'\{m(\log 10)\} = c$

$$(31.31)\qquad \sum_{\lambda_n>10}^{\infty} e^{\lambda_n\phi'\{m(\log\lambda_n)\}-\lambda_n x} \leqq \sum_{c>\phi'\{m(\log\lambda_n)\}\geqq x/2} e^{\lambda_n\phi'\{m(\log\lambda_n)\}} + \sum_{n=1}^{\infty} e^{-\lambda_n x/2}.$$

But $\phi'\{m\ (\log\lambda_n)\} \geqq \frac{1}{2}x$ is equivalent to $m(\log\lambda_n) \leqq \Phi(\frac{1}{2}x)$. Since $m(u)$ is the inverse function of $\phi(u)$, this is equivalent to $\log\lambda_n \leqq \phi(\Phi(\frac{1}{2}x))$ or

$$(31.32)\qquad \lambda_n \leqq e^{\phi(\Phi(x/2))}.$$

But

$$\phi(\Phi(\tfrac{1}{2}x)) = \int_0^{\Phi(x/2)} \phi'(y)\,dy.$$

Setting $\phi'(y) = z$,

$$\phi(\Phi(\tfrac{1}{2}x)) = -\int_{/2}^{\phi'(0)} z\Phi'(z)\,dz = -\int_{/2}^{\phi'(0)} \mu_4'(2z)\,dz$$

$$= \tfrac{1}{2}\mu_4(x) - \tfrac{1}{2}\mu_4(2\phi'(0)).$$

Thus for small values of x, (31.32) becomes

$$\lambda_n < e^{(3/4)\mu_4(x)}$$

and therefore (31.31) becomes for small x

$$(31.33)\qquad \sum_{\lambda_n>10}^{\infty} e^{\lambda_n\phi'\{m(\log\lambda_n)\}-\lambda_n x} \leqq \sum_{\lambda_n\leqq e^{(3/4)\mu_4(x)}} e^{ce^{(3/4)\mu_4(x)}} + \sum_{n=1}^{\infty} e^{-(1/2)\lambda_n x}.$$

Clearly for small x,

$$\sum_{n=1}^{\infty} e^{(-1/2)\lambda_n x} = O\left(\frac{1}{x}\right).$$

Also the number of $\lambda_n \leqq e^{(3/4)\mu_4(x)}$ is of the order of $e^{(3/4)\mu_4(x)}$. Thus (31.33) becomes

$$\sum_{\lambda_n > 10}^{\infty} e^{\lambda_n \phi'[m(\log \lambda_n)] - \lambda_n x} \leqq c_{33} e^{(3/4)\mu_4(x)} e^{ce^{(3/4)\mu_4(x)}} \leqq e^{e^{\mu_4(x)}}$$

for small x. But $\mu_4(x) \leqq \mu(x)$. Thus the proof of the lemma is completed.

Proof of Theorem XLVIII. This is an immediate consequence of Lemmas 29.1, 31.1, and 31.2 once we prove that $\{\lambda_n\}$ as defined in Lemma 31.1 has density $D > 0$. If $\Lambda(u)$ is the number of $\lambda_n < u$, clearly from Lemma 31.1,

$$\Lambda(u) = \sum_{n \leqq m(\log u)} 2\delta\phi'(n)e^{\phi(n)} + O(u\phi'[m(\log u)])$$

where $m(u)$ is the inverse function of $\phi(u)$.

From this

$$\Lambda(u) = 2\delta \int_0^{m(\log u)} \phi'(y)e^{\phi(y)}\, dy + O(u\phi'[m(\log u)])$$

$$= 2\delta u + O(u\phi'[m(\log u)]).$$

Since $\phi'(u) = o(1)$,

$$\lim_{u \to \infty} \frac{\Lambda(u)}{u} = 2\delta = D > 0.$$

CHAPTER X

THE GENERAL HIGHER INDICES THEOREM[1]

32. Introduction. The following Tauberian theorem is due to Hardy and Littlewood.[2]

Let $\{\mu_k\}$ *be a sequence of increasing positive numbers such that*

$$\frac{\mu_{k+1}}{\mu_k} \geqq \theta > 1, \qquad k = 1, 2, \cdots,$$

let

$$\sum_1^\infty a_k e^{-\mu_k x} = f(x)$$

converge for $x > 0$, *and let*

$$\lim_{x \to +0} f(x) = s.$$

Then

$$\sum_1^\infty a_k = s.$$

This theorem differs from the usual Tauberian theorem in that there is no restriction on the size of the coefficients a_k. The proof of this theorem that is of most interest to us here is due to Wiener.[3] This proof is based on the use of biorthogonal functions which will be used here.

This theorem of Hardy and Littlewood, the higher indices theorem, can be stated in different form. If we set $\log \mu_k = \lambda_k$ and assume that $\mu_1 > 1$, then the higher indices theorem becomes:

Let $\{\lambda_k\}$ *be a sequence of increasing positive numbers such that*

$$\lambda_{k+1} - \lambda_k \geqq L > 0, \qquad k = 1, 2, \cdots.$$

Let

$$\sum_1^\infty a_k \int_{\lambda_k - x}^\infty e^{-e^y} e^y \, dy = f(x) \tag{32.01}$$

converge for any $x < \infty$. *If*

[1] Cf. Levinson, *General gap Tauberian theorems*, Proceedings of the London Mathematical Society, (2), vol. 44 (1938), p. 289.

[2] Hardy and Littlewood, *Abel's theorem and its converse* (II), Proceedings of the London Mathematical Society, (2), vol. 22 (1924), p. 254.

[3] N. Wiener, *A Tauberian gap theorem of Hardy and Littlewood*, Journal of the Chinese Mathematical Society, vol. 1 (1936), p. 15.

$$\lim_{x\to\infty} f(x) = s,$$

then

$$\sum_1^\infty a_k = s.$$

We shall generalize this result of Hardy and Littlewood by considering

$$f(x) = \sum_1^\infty a_n \int_{\lambda_n - x}^\infty K(y)\, dy$$

in place of (32.01) for a wide class[4] of $K(x)$. In case the a_n are suitably restricted in size, Wiener showed that the only necessary restriction on $K(x)$ for a Tauberian theorem is that its Fourier transform should not vanish at any point. That this condition is not sufficient for a higher indices theorem where the a_n are unrestricted we shall prove in the following theorem.

THEOREM L. *If* $\lambda_n = n$ *and*

$$a_n = \frac{(-1)^n e^{n^2/2}}{n!}, \tag{32.02}$$

then

$$\lim_{x\to\infty} \sum_{n=1}^\infty a_n \int_{\lambda_n - x}^\infty e^{-y^2/2}\, dy = s \tag{32.03}$$

exists, but

$$\sum_1^\infty a_n \tag{32.04}$$

does not converge.

This theorem shows that $K(x) = e^{-x^2/2}$, even though its Fourier transform $e^{-u^2/2}$ does not vanish at any point, cannot be included in the class of $K(x)$ for which

$$\lim_{x\to\infty} \sum_1^\infty a_n \int_{\lambda_n - x}^\infty K(y)\, dy = s, \qquad \lambda_{n+1} - \lambda_n \geqq L > 0,$$

implies

$$\sum_1^\infty a_n = s \Big/ \int_{-\infty}^\infty K(y)\, dy.$$

Proof of Theorem L. Since

$$e^{-e^{-y}} = \sum_0^\infty \frac{(-1)^n}{n!} e^{-ny},$$

[4] In the case of ordinary Tauberian theorems, that is, where the a_n are restricted in magnitude, this generalization was made by Wiener. See, for example, *A new method in Tauberian theorems*, Journal of Mathematics and Physics, vol. 7 (1928), p. 161.

we obtain

$$e^{-y^2/2}e^{-e^{-y}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} e^{-ny-y^2/2}.$$

Thus

$$\begin{aligned}\int_{-x}^{\infty} e^{-y^2/2}e^{-e^{-y}}\,dy &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-x}^{\infty} e^{-ny-y^2/2}\,dy \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n e^{n^2/2}}{n!} \int_{-x}^{\infty} e^{-(y+n)^2/2}\,dy \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n e^{n^2/2}}{n!} \int_{n-x}^{\infty} e^{-y^2/2}\,dy.\end{aligned}$$

Let

$$s = \int_{-\infty}^{\infty} e^{-y^2/2}e^{-e^{-y}}\,dy - \int_{-\infty}^{\infty} e^{-y^2/2}\,dy.$$

Then

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} \frac{(-1)^n e^{n^2/2}}{n!} \int_{n-x}^{\infty} e^{-y^2/2}\,dy = s.$$

This proves (32.03). Since as $n \to \infty$

$$\frac{e^{n^2/2}}{n!} \to \infty,$$

it follows that (33.04) cannot converge. This completes the proof.

Thus for $K(x) = e^{-x^2/2}$ with Fourier transform $k(u) = e^{-u^2/2}$ the higher indices theorem is not true, but for $K(x) = e^{-e^x}e^x$ with Fourier transform $\Gamma(1 - iu)$ it is true. Therefore a general higher indices theorem will be sufficiently restrictive to distinguish between these two $K(x)$.

In what follows we shall assume that

$$\int_{-\infty}^{\infty} K(x)\,dx = 1.$$

The basic result of this chapter is:

THEOREM LI. *Let*

(32.05) $$\sum_{1}^{\infty} a_n \int_{\lambda_n - x}^{\infty} K(y)\,dy = f(x)$$

converge uniformly for all $x \leq X$ for any X. Let

(32.06) $$\lambda_{n+1} - \lambda_n \geqq L > 0, \qquad n \geqq 1.$$

Let $K(x) \in L(-\infty, \infty)$. Its Fourier transform is

$$k(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} K(x) e^{-iux}\, dx.$$

Let $k(u + iv)$ be an analytic function of $w = u + iv$, in the half-plane $v > 0$ and continuous for $v \geqq 0$, and let $k'(u)$ exist. Let $c = 2\pi/L$ and let A be a positive constant. Suppose that ξ is a real variable. Let

$$\max_{|\xi| \leq c} \left| \frac{k(u + \xi)}{k(u)} \right| \leqq e^{\theta(u)}, \tag{32.07}$$

$$\max_{|\xi| \leq c} \left| \frac{k(w + \xi)}{k(w)} \right| \leqq A e^{A|w|}, \qquad v \geqq 0, \tag{32.08}$$

$$\left| \frac{k'(u)}{k(u)} \right| \leqq e^{\theta(u)}, \tag{32.09}$$

where $\theta(u)$ is a positive even function of u, monotone increasing for $u > 0$, and

$$\int_1^{\infty} \frac{\theta(u)}{u^2}\, du < \infty. \tag{32.10}$$

Then

$$\limsup_{n \to \infty} |a_n| \leqq C_0 \limsup_{x \to \infty} |f(x)| \tag{32.11}$$

where C_0 is a constant depending only on $K(x)$ and L.

This theorem reduces the higher indices to the level of an ordinary Tauberian theorem. Using Wiener's general Tauberian theorem, a corollary of Theorem LI is that *if*

$$\lim_{x \to \infty} f(x) = s, \tag{32.12}$$

then

$$\sum_1^{\infty} a_k = s. \tag{32.13}$$

Here we shall prove a more restricted corollary.

THEOREM LII. *Let the hypothesis of Theorem* LI *be satisfied. In addition let $xK(x) \in L(-\infty, \infty)$. Then* (32.12) *implies* (32.13).

The statement of Theorem LII includes the Hardy-Littlewood higher indices theorem since

$$k(w) = \frac{1}{(2\pi)^{1/2}} \Gamma(1 - iw)$$

by well-known theorems satisfies (32.07), (32.08) and (32.09).

Let us now see what excludes $K(x) = e^{-x^2/2}/(2\pi)^{1/2}$. Here $k(w) = e^{-u^2/2}/(2\pi)^{1/2}$

and clearly (32.08) and (32.09) are satisfied. As regards (32.07), for large u

$$\max_{|\xi|\leqq c} \frac{e^{-(u+\xi)^2/2}}{e^{-u^2/2}} = e^{c|u|-c^2/2}$$

and therefore here $\theta(u) = |u|$. Thus (32.07) is not satisfied and it is this that excludes $e^{-x^2/2}/(2\pi)^{1/2}$. *This shows at once that requirement* (32.07) *is essential.* Once (32.07) is assumed, (32.08) is implied by what are apparently much weaker restrictions, but this is of little interest since for the usual $K(x)$, (32.07) will assure (32.08) anyway. As for (32.09), it is no restriction at all for the usual $K(x)$. Under very general conditions (32.07) will in fact assure the much more stringent

$$\left| \frac{k'(u)}{k(u)} \right| \leqq \theta(u).$$

Thus (32.07) is the basic criterion for Theorems LI and LII.

Clearly (32.09) implies *that* $k(w) \neq 0$ *in the half-plane*, $v \geqq 0$. It can be shown that in fact $k(w)$ can have a finite number of zeros in the half-plane $v \geqq 0$, so long as $k(u) \neq 0$, ($|u| \leqq \pi/L$), by varying the proof given here for the case where there are no zeros.

An alternative statement of Theorem LII is

THEOREM LIII. *Let* $\{\mu_n\}$ *be an increasing sequence such that*

$$\mu_{n+1}/\mu_n \geqq e^L, \qquad L > 0.$$

Let

$$\sum_1^\infty a_n N(x\mu_n) = f(x)$$

converge uniformly for $x \geqq x_0$ *for any* $x_0 > 0$. *Let* $N'(x) \in L(0, \infty)$ *and* $N'(x) \cdot \log x \in L(0, \infty)$. *Let*

$$k(u) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty N'(x) x^{-iu}\, dx.$$

If $k(u + iv)$ *satisfies* (32.07), (32.08) *and* (32.09), *then*

$$\lim_{x\to+0} f(x) = s$$

implies that

$$\sum_1^\infty a_k = s/N(0).$$

A particular case of this theorem is $N(x) = xe^{-x}/(1 - e^{-x})$, for which

$$k(u) = \frac{iu}{(2\pi)^{1/2}} \Gamma(1 - iu)\zeta(1 - iu).$$

That $k(w)$ satisfies (32.07), (32.08), and (32.09) follows from well-known

theorems on the gamma and zeta functions. A series

$$\sum_{1}^{\infty} a_n \frac{\mu_n e^{-\mu_n x}}{1 - e^{-\mu_n x}}$$

is known as a Lambert series. Thus a corollary of Theorem LIII is that *the higher indices theorem holds for Lambert series.* This result was unknown before the proof of Theorem LII in 1937. In fact the Hardy-Littlewood higher indices theorem remained an essentially isolated theorem until the theorem for $K(x)$ was proved.

We shall also prove the following result.

THEOREM LIV.[5] *Let the hypothesis of Theorem LI be satisfied. In addition let $xK(x) \in L(-\infty, \infty)$. Then*

$$\liminf_{x\to\infty} \{f(x) - s\} = -\Omega, \qquad \limsup_{x\to\infty} \{f(x) - s\} = \Omega \tag{32.14}$$

implies that

$$\limsup_{n\to\infty} \left| \sum_{1}^{n} a_k - s \right| \leqq C_1\Omega \tag{32.15}$$

where C_1 is an absolute constant depending only on L and $K(x)$. In particular, if $\lambda_{n+1} - \lambda_n \to \infty$ then $C_1 = 1$.

33. Reduction to a lemma on biorthogonal functions. The proof of Theorem LI can be at once reduced to the proof of the following lemma. $C_2, C_3, \cdots$ denote positive constants depending only on L and $K(x)$.

LEMMA 33 1. *If the hypothesis of Theorem LI is satisfied, there exists a sequence of functions $\{R_n(x, B)\}$, $(n \geqq 1)$, such that*

$$\int_{-\infty}^{\infty} |R_n(x, B)| \, dx < C_2, \tag{33.01}$$

$$R_n(x, B) = 0, \qquad x > C_3B, \tag{33.02}$$

$$\int_{-\infty}^{\lambda_n/2} |R_n(x, B)| \, dx \leqq \frac{C_4}{\lambda_n^{1/2}}, \qquad n \geqq 1, \tag{33.03}$$

$$\int_{-\infty}^{\infty} R_n(x, B) \, dx \int_{\lambda_k - x}^{\infty} K(y) \, dy = 0, \qquad k \neq n, \tag{33.04}$$

and

$$\lim_{B\to\infty} \int_{-\infty}^{\infty} R_n(x, B) \, dx \int_{\lambda_n - x}^{\infty} K(y) \, dy = 1. \tag{33.05}$$

Before proving the lemma we shall use it to obtain Theorem LI.

[5] For $K(x) = e^{-e^x}e^x$ this theorem was proved by Ingham, *On the higher indices theorem of Hardy and Littlewood*, Quarterly Journal of Mathematics, vol 8 (1937), p. 1.

Proof of Theorem LI. Clearly

$$\int_{-\infty}^{\infty} R_n(x, B)f(x)\,dx = \int_{-\infty}^{\infty} R_n(x, B)\,dx \sum_{k=1}^{\infty} a_k \int_{\lambda_k - x}^{\infty} K(y)\,dy.$$

Since $R_n(x, B) = 0$ for $x > C_3B$ and since the series on the right converges uniformly for $x < C_3B$, integration and summation can be inverted to give

$$\int_{-\infty}^{\infty} R_n(x, B)f(x)\,dx = \sum_{k=1}^{\infty} a_k \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_k - x}^{\infty} K(y)\,dy.$$

By the orthogonality property, (33.04), this becomes

$$a_n \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_n - x}^{\infty} K(y)\,dy = \int_{-\infty}^{\infty} R_n(x, B)f(x)\,dx. \tag{33.06}$$

By (33.01) and (33.03)

$$\left| \int_{-\infty}^{\infty} R_n(x, B)f(x)\,dx \right| \leqq \max |f(x)| \int_{-\infty}^{\lambda_n/2} |R_n(x, B)|\,dx$$

$$+ \max_{x \geqq \lambda_n/2} |f(x)| \int_{\lambda_n/2}^{\infty} |R_n(x, B)|\,dx$$

$$\leqq \frac{C_4}{\lambda_n^{1/2}} \max |f(x)| + C_2 \max_{x \geqq \lambda_n/2} |f(x)|.$$

Thus (33.06) becomes

$$\left| a_n \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_n - x}^{\infty} K(y)\,dy \right| \leqq \frac{C_4}{\lambda_n^{1/2}} \max |f(x)| + C_2 \max_{x \geqq \lambda_n/2} |f(x)|.$$

By (33.05), letting $B \to \infty$, this becomes

$$|a_n| \leqq \frac{C_4}{\lambda_n^{1/2}} \max |f(x)| + C_2 \max_{x \geqq \lambda_n/2} |f(x)|.$$

From this (32.11) follows at once, showing that Theorem LI is an immediate consequence of Lemma 33.1.

Theorem LII can now easily be proved by an argument like that originally used by Tauber.

Proof of Theorem LII. There is obviously no restriction in assuming that $s = 0$; that is,

$$\lim_{x \to \infty} f(x) = 0.$$

By (32.11) it follows that

$$\lim_{n \to \infty} a_n = 0. \tag{33.07}$$

Let us now take

$$\nu_n = \tfrac{1}{2}(\lambda_n + \lambda_{n+1}).$$

Then

$$\sum_1^n a_k \int_{\lambda_k-\nu_n}^{\infty} K(y)\,dy + \sum_{n+1}^{\infty} a_k \int_{\lambda_k-\nu_n}^{\infty} K(y)\,dy = f(\nu_n).$$

This can be written as

$$(33.08)\qquad \sum_1^n a_k - \sum_1^n a_k \int_{-\infty}^{\lambda_k-\nu_n} K(y)\,dy + \sum_{n+1}^{\infty} a_k \int_{\lambda_k-\nu_n}^{\infty} K(y)\,dy = f(\nu_n).$$

Clearly

$$\begin{aligned}
&\left|\sum_1^n a_k \int_{-\infty}^{\lambda_k-\nu_n} K(y)\,dy\right| \\
&\quad\leqq \sum_1^{[n/2]} |a_k| \int_{-\infty}^{\lambda_k-\nu_n} |K(y)|\,dy + \sum_{[n/2]+1}^{n} |a_k| \int_{-\infty}^{\lambda_k-\nu_n} |K(y)|\,dy \\
&\quad\leqq \max |a_k| \sum_{k=1}^{[n/2]} \int_{-\infty}^{-(n-k)L} |K(y)|\,dy + \max_{k\geq n/2} |a_k| \sum_{[n/2]}^{n} \int_{-\infty}^{-(n+1/2-k)L} |K(y)|\,dy \\
&\quad\leqq \max |a_k|\, n \int_{-\infty}^{-nL/2} |K(y)|\,dy + \max_{k\geq n/2} |a_k| \sum_{j=0}^{\infty} \int_{-\infty}^{(1/2+j)L} |K(y)|\,dy \\
&\quad\leqq \max |a_k|\, n \int_{-\infty}^{nL/2} |K(y)|\,dy + \max_{k\geq n/2} |a_k| \sum_{j=0}^{\infty} (j+1) \int_{-(3/2+j)L}^{-(1/2+j)L} |K(y)|\,dy \\
&\quad\leqq \max |a_k| \frac{2}{L} \int_{-\infty}^{-nL/2} |yK(y)|\,dy + \max_{k\geqq n/2} |a_k| \frac{2}{L} \sum_{j=0}^{\infty} \int_{-(3/2+j)L}^{-(1/2+j)L} |yK(y)|\,dy.
\end{aligned}$$

From this it follows at once that

$$(33.09)\qquad \begin{aligned}
&\left|\sum_1^n a_k \int_{-\infty}^{\lambda_k-\nu_n} K(y)\,dy\right| \\
&\qquad \leqq \max |a_k| \frac{2}{L} \int_{-\infty}^{-nL/2} |yK(y)|\,dy + \max_{k\geqq n/2} |a_k| \frac{2}{L} \int_{-\infty}^{-L/2} |yK(y)|\,dy.
\end{aligned}$$

Also

$$(33.10)\qquad \begin{aligned}
\left|\sum_{n+1}^{\infty} a_k \int_{\lambda_k-\nu_n}^{\infty} K(y)\,dy\right| &\leqq \max_{k>n} |a_k| \sum_{j=0}^{\infty} \int_{(1/2+j)L}^{\infty} |K(y)|\,dy \\
&\leqq \max_{k>n} |a_k| \sum_{j=0}^{\infty} (j+1) \int_{(1/2+j)L}^{(3/2+j)L} |K(y)|\,dy \\
&\leqq \max_{k>n} |a_k| \frac{2}{L} \int_{L/2}^{\infty} |yK(y)|\,dy.
\end{aligned}$$

From (33.08), (33.09), and (33.10) it follows that

(33.11)
$$\left|\sum_1^n a_k - f(\nu_n)\right| \leqq \max |a_k| \frac{2}{L}\int_{-\infty}^{-nL/2} |yK(y)|\,dy + \max_{k\geqq n/2} |a_k| \frac{2}{L}\int_{-\infty}^{\infty} |yK(y)|\,dy.$$

By (33.07) and the integrability of $yK(y)$ it follows that

$$\sum_1^\infty a_k = \lim_{n\to\infty} f(\nu_n) = s.$$

This completes the proof of Theorem LII.

Proof of Theorem LIV. There is no restriction in assuming that $s = 0$ since, for example, a_1 can be replaced by $a_1 - s$. Thus (32.14) becomes

(33.12)
$$\liminf_{x\to\infty} f(x) = -\Omega, \limsup_{x\to\infty} f(x) = \Omega.$$

It follows from (32.11) that

$$\limsup_{n\to\infty} |a_n| \leqq C_0\Omega.$$

Used in (33.11) this gives

(33.13)
$$\lim_{n\to\infty}\left|\sum_1^n a_k - f(\nu_n)\right| \leqq \frac{2C_0\Omega}{L}\int_{-\infty}^{\infty} |yK(y)|\,dy,$$

from which (32.15) follows easily.

We now consider the case in which $\lambda_{n+1} - \lambda_n \to \infty$. For $N > 0$, let

$$\sum_{N+1}^\infty a_k \int_{\lambda_k - x}^\infty K(y)\,dy = f_N(x), \qquad L_N = \min_{k>N} (\lambda_{k+1} - \lambda_k).$$

Applying (33.13) to $f_N(x)$,

$$\limsup_{n\to\infty}\left|\sum_{N+1}^n a_k - f_N(\nu_n)\right| \leqq \frac{2C_0\Omega}{L_N}\int_{-\infty}^{\infty} |yK(y)|\,dy.$$

But

$$\lim_{n\to\infty}\left|f(\nu_n) - f_N(\nu_n) - \sum_1^N a_k\right| = 0.$$

Thus

$$\limsup_{n\to\infty}\left|\sum_1^n a_k - f(\nu_n)\right| \leqq \frac{2C_0\Omega}{L_N}\int_{-\infty}^{\infty} |yK(y)|\,dy.$$

Taking N sufficiently large we can make L_N arbitrarily large. Thus

$$\lim_{n\to\infty}\left|\sum_1^n a_k - f(\nu_n)\right| = 0.$$

Therefore

$$\limsup_{n\to\infty} \left| \sum_1^n a_k - s \right| \leqq \limsup_{n\to\infty} |f(\nu_n) - s| + \lim_{n\to\infty} \left| \sum_1^n a_k - f(\nu_n) \right| = \Omega.$$

This proves $C_1 = 1$ when $\lambda_{n+1} - \lambda_n \to \infty$.

34. Proof of the lemma. We now turn to the proof of Lemma 33.1. We will require several auxiliary results. The first of these is an interpolation result.

LEMMA 34.1. *For every $n > 0$ there exists an $H_n(s) \in L(-\infty, \infty)$, such that*

$$H_n(\lambda_n) = 1; \qquad H_n(\lambda_k) = 0, \qquad k \neq n, \tag{34.01}$$

where $\{\lambda_n\}$ are as in Theorem LI. *If*

$$h_n(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H_n(s) e^{-isu}\, ds, \tag{34.02}$$

then $h_n(u) = 0$, $|u| > c$, and

$$h_n(u) = g_n(u) e^{-iu\lambda_n}, \tag{34.03}$$

where

$$|g_n(u)| < C_5, \qquad |g_n'(u)| < C_6. \tag{34.04}$$

Proof of Lemma 34.1. Throughout this proof n is some fixed positive integer. For $-\infty < k < \infty$ let

$$\sigma_k = \tfrac{1}{2}kL \text{ if } |\tfrac{1}{2}kL - \lambda_m + \lambda_n| > \tfrac{1}{4}L \text{ for all } m > 0,$$

$$\sigma_k = \tfrac{1}{2}kL \text{ if } \tfrac{1}{2}kL - \lambda_m + \lambda_n = \tfrac{1}{4}L \text{ for some } m,$$

$$\sigma_k = \lambda_m - \lambda_n \text{ if } -\tfrac{1}{4}L \leqq \tfrac{1}{2}kL - \lambda_m + \lambda_n < \tfrac{1}{4}L.$$

This defines $\{\sigma_k\}$ uniquely and

$$|\sigma_k - \tfrac{1}{2}kL| \leqq \tfrac{1}{4}L.$$

Also

$$\sigma_{k+1} - \sigma_k \geqq \tfrac{1}{4}L.$$

Moreover $\{\sigma_k\}$ includes $\{\lambda_m - \lambda_n\}$ for all $m \neq n$.

Let

$$T(s) = \prod_1^{\infty} \left(1 - \frac{s}{\sigma_k}\right)\left(1 - \frac{s}{\sigma_{-k}}\right).$$

Then

$$\left| \frac{T(s)}{\sin 2\pi s/L} \right| = \frac{L}{2\pi|s|} \prod_1^{\infty} \left| \frac{1 - s/\sigma_k}{1 - 2s/kL} \right| \left| \frac{1 - s/\sigma_{-k}}{1 + 2s/kL} \right|.$$

Clearly

$$\left|\frac{1 - s/\sigma_k}{1 - 2s/kL}\right| \leqq 1 + \left|\frac{s(\sigma_k - \frac{1}{2}kL)}{\sigma_k(s - \frac{1}{2}kL)}\right| \leqq 1 + \frac{\frac{1}{4}L|s|}{(\frac{1}{2}|k|L - \frac{1}{4}L)|s - \frac{1}{2}kL|}$$

$$= 1 + \frac{|s|}{(2|k| - 1)|s - \frac{1}{2}kL|} \leqq e^{|s/k(s-kL/2)|}.$$

If N is defined by $\frac{1}{2}NL - \frac{1}{4}L \leqq |s| < \frac{1}{2}NL + \frac{1}{4}L$, if s is in the right half-plane, and if $\sum''$ indicates the omission of $k = 0$ and $k = N$, then for $|s| > 10L$

$$\sum{}'' \frac{1}{|k(s - \frac{1}{2}kL)|} \leqq \frac{2}{L}\sum_{-\infty}^{-N}\frac{1}{k^2} + \frac{1}{|s|}\sum_{-N+1}^{-1}\frac{1}{|k|} + \frac{3}{|s|}\sum_{1}^{[N/2]}\frac{1}{k}$$

$$+ \frac{4}{N}\sum_{[N/2]+1}^{N-1}\frac{1}{(N - n - \frac{1}{2})L} + \frac{2}{N}\sum_{N+1}^{2N}\frac{1}{(n - N - \frac{1}{2})L}$$

$$+ \frac{2}{L}\sum_{2N+1}^{\infty}\frac{1}{(n - N - \frac{1}{2})^2}$$

$$\leqq \frac{100 \log |s|}{|s|}.$$

This must hold just as well for s in the left half-plane. Thus for $|s| > 10L$, by the three results above,

$$\left|\frac{T(s)}{\sin 2\pi s/L}\right| \leqq 100L\left|\frac{s - \sigma_N}{s - \frac{1}{2}NL}\right||s|^{100}.$$

It follows easily from this that

$$|T(s)| \leqq C_7(1 + |s|)^{100}e^{2\pi|t|/L} \tag{34.05}$$

where $t = \Im s$.

From the definition of σ_k it follows that in an interval of length $500L$ there are at least $200\sigma_k$ not of the form $\lambda_m - \lambda_n$, $(m > 0)$ Let such σ_k in $(-250L, 250L)$ be denoted by $\tau_1, \tau_2, \cdots, \tau_{200}$. Then by (34.05),

$$\left|\frac{T(s)}{(1 - s/\tau_1)\cdots(1 - s/\tau_{200})}\right| \leqq \frac{C_8}{(1 + |s|)^4}e^{2\pi|t|/L}.$$

Let

$$H_n(s) = \frac{T(s - \lambda_n)}{\left(1 - \frac{s - \lambda_n}{\tau_1}\right)\cdots\left(1 - \frac{s - \lambda_n}{\tau_{200}}\right)}.$$

Then $H_n(s)$ satisfies (34.01). Also

$$|H_n(s)| \leqq \frac{C_8}{(1 + |s - \lambda_n|)^4}e^{c|t|}, \tag{34.06}$$

where we recall that $c = 2\pi/L$. If

$$g_n(u) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} H_n(s + \lambda_n)e^{-isu}\,ds,$$

then by (34.06)

$$| g_n(u) | < C_9 .$$

Similarly

$$| g_n'(u) | < C_{10} .$$

By the Cauchy integral theorem and (34.06) the path of integration for $g_n(u)$ can be closed in the lower half-plane if $u > c$ giving $g_n(u) = 0$. A similar result holds for $u < -c$. Thus

$$| g_n(u) | = 0, \qquad | u | > c.$$

If $h_n(u)$ is defined as in (34.02), then clearly

$$h_n(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H_n(s + \lambda_n) e^{-iu(s+\lambda_n)} ds = e^{-iu\lambda_n} g_n(u).$$

This completes the proof of the lemma.

LEMMA 34.2. *There exists a function $\phi(u + iv)$ analytic in the upper half-plane $v \geqq 0$ and such that*

(34.07) $$| \phi(u + iv) | \leqq C_{12} e^{C_{11} v}, \qquad v \geqq 0,$$

$| \phi(u) |$ is even, and

(34.08) $$| \phi(u) | \leqq C_{13} \frac{e^{-2\theta(u)}}{1 + u^4},$$

where $\theta(u)$ satisfies the conditions given in Theorem LI. *Moreover, if*

(34.09) $$\Phi(s) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(u) e^{isu} du,$$

then

(34.10) $$| \Phi(s) | \leqq C_{14} , \qquad | \Phi'(s) | \leqq C_{15} ,$$

and

(34.11) $$\Phi(0) = 1.$$

In this lemma s is a real variable.

Proof of Lemma 34.2. Let $\theta_1(u)$ be an even function of u defined for $u > 0$ by

$$\theta_1(u) = 4\theta(u) + 2u^{1/2}.$$

Clearly $\theta_1(u)$ is monotone increasing for $u > 0$. Let

$$\lambda(u, v) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(v+1)\theta_1(\xi)}{(u-\xi)^2+(v+1)^2}\,d\xi = \Re\frac{1}{\pi}\int_{-\infty}^{\infty}\theta_1(\xi)\frac{i}{u-\xi+i(v+1)}\,d\xi.$$

By (32.10) the above integral defines $\lambda(u, v)$ as an harmonic function in the half-plane $v > -1$. For $u > 0$

$$\lambda(u, 0) \geqq \frac{1}{\pi}\int_{u}^{\infty}\frac{\theta_1(\xi)}{(u-\xi)^2+1}\,d\xi \geqq \frac{\theta_1(u)}{\pi}\int_0^{\infty}\frac{d\xi}{1+\xi^2}$$
$$\geqq \tfrac{1}{2}\theta_1(u) \geqq 2\theta(u) + u^{1/2}.$$

Since $\lambda(u, 0)$ is even, it follows that, for all u,

$$\lambda(u, 0) \geqq 2\theta(u) + |u|^{1/2}. \tag{34.12}$$

It is well-known that there exists a function $\mu(u, v)$ conjugate to $\lambda(u, v)$. Let $w = u + iv$. Then

$$g(w) = \lambda(u, v) + i\mu(u, v)$$

is analytic in the half-plane $v > -1$. Since $\lambda(u, v) \geqq 0$,

$$|e^{-g(w)}| \leqq 1. \tag{34.13}$$

Along the real axis, by (34.12)

$$|e^{-g(u)}| \leqq e^{-2\theta(u)-|u|^{1/2}}. \tag{34.14}$$

Let

$$\Phi_1(s) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-g(u)}e^{isu}\,du.$$

$\Phi_1(s)$ is not identically zero since it is the Fourier transform of $e^{-g(u)}$. Thus for some value of s, $\Phi_1(s) \neq 0$. Let such a value of s be a. Then

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-g(u)}e^{iau}\,du = b \neq 0. \tag{34.15}$$

Let

$$\phi(w) = \frac{e^{-g(w)+iaw}}{b}.$$

Then by (34.13), (34.07) follows, and by (34.14), (34.08) follows. Let $\Phi(s)$ be defined as in (34.09). Then by (34.15), (34.11) is satisfied. (34.10) follows easily from (34.08). This completes the proof of the lemma.

Proof of Lemma 33.1. Using the terminology in the statements of Theorem LI and Lemmas 34.1 and 34.2, let us define

$$r_n(u, B) = -\frac{iuBe^{-iu\lambda_n}}{(2\pi)^{1/2}k(u)}\int_{u-c}^{u+c} k(y)\phi(By)g_n(u-y)e^{i\lambda_n y}\,dy \tag{34.16}$$

By (32.07) and (34.04)

$$|r_n(u, B)| \leqq C_5 B |u| e^{\theta(u)} \int_{u-c}^{u+c} |\phi(By)|\, dy \leqq C_5 |u| e^{\theta(u)} \int_{B(u-c)}^{B(u+c)} |\phi(y)|\, dy.$$

Thus for $|u| \leqq 2c$ it follows from (34.08) that

$$|r_n(u, B)| \leqq C_5 |u| e^{\theta(2c)} \int_{-\infty}^{\infty} |\phi(y)|\, dy \leqq C_{16} |u|. \tag{34.17}$$

For $|u| > 2c$ and $B > 2$ it follows from (34.08) that

$$|r_n(u, B)| \leqq C_5 |u| e^{\theta(u)} \int_{|u|}^{\infty} \phi(y)\, dy \leqq \frac{C_{17}}{1 + u^2}. \tag{34.18}$$

Thus

$$\int_{-\infty}^{\infty} |r_n(u, B)|\, du < C_{18}. \tag{34.19}$$

From (34.16) we also have

$$\frac{d}{du}\{e^{iu\lambda_n} r_n(u, B)\} = -\frac{iuB}{(2\pi)^{1/2} k(u)} \int_{u-c}^{u+c} k(y)\phi(By) g_n'(u - y) e^{i\lambda_n y}\, dy$$
$$- \frac{iB}{(2\pi)^{1/2} k(u)} \left(1 - \frac{uk'(u)}{k(u)}\right) \int_{u-c}^{u+c} k(y)\phi(By) g_n(u - y) e^{i\lambda_n y}\, dy.$$

Treating each of the terms on the right in a manner similar to that used on (34.16), it follows easily that

$$\left|\frac{d}{du}\{e^{iu\lambda_n} r_n(u, B)\}\right| \leqq \frac{C_{19}}{1 + u^2}. \tag{34.20}$$

If we define

$$R_n(x, B) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} r_n(u, B) e^{iux}\, du,$$

it follows from (34.19) that

$$|R_n(x, B)| < C_{18}. \tag{34.21}$$

Also

$$R_n(x + \lambda_n, B) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} r_n(u, B) e^{iu\lambda_n} e^{iux}\, du,$$

and integrating by parts,

$$-ixR_n(x + \lambda_n, B) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{d}{du}\{r_n(u, B) e^{iu\lambda_n}\} e^{iux}\, du.$$

By the theorem for Fourier transforms, Theorem E, and (34.20)

$$\int_{-\infty}^{\infty} x^2 |R_n(x + \lambda_n, B)|^2 dx \leqq C_{20}. \tag{34.22}$$

From this and (34.21) it follows by the Schwarz inequality that

$$\int_{-\infty}^{\infty} |R_n(x, B)| \, dx < C_2.$$

This is (33.01). Again by the Schwarz inequality

$$\int_{-\infty}^{\lambda_n/2} |R_n(x, B)| \, dx \leqq \left\{ \int_{-\infty}^{\infty} (x - \lambda_n)^2 |R_n(x, B)|^2 dx \right\}^{1/2} \left\{ \int_{-\infty}^{\lambda_n/2} \frac{dx}{(x - \lambda_n)^2} \right\}^{1/2}.$$

Using (34.22) we have (33.03) at once.

Clearly by (34.16) for real w

$$r_n(w, B) = \frac{-iBw}{(2\pi)^{1/2} k(w)} \int_{-c}^{c} k(w - y)\phi\{B(w - y)\} h_n(y) \, dy.$$

But this definition shows that $r_n(w, B)$ is analytic in the half-plane $v > 0$ and continuous in $v \geqq 0$. For $v \geqq 0$, by (32.08), (34.07), and (34.04)

$$|r_n(w, B)| \leqq B \, |w| \, Ae^{A|w|} C_{12} e^{C_{11} Bv} 2cC_5 \, .$$

Or for $B > 2$

$$|r_n(w, B)| \leqq BC_{21} e^{BC_{22}|w|}, \qquad v \geqq 0. \tag{34.23}$$

By Theorems C′ and C of Phragmén-Lindelöf, (34.18) and (34.23) imply that in the upper half-plane $v \geqq 0$

$$|r_n(w, B) e^{2BC_{22}iw}| \leqq \frac{C_{23}}{1 + |w|^2}.$$

From the definition of $R_n(x, B)$, it now follows that for $x > 2BC_{22}$ the path of integration can be closed in the upper half-plane giving

$$R_n(x, B) = 0, \qquad x > 2BC_{22},$$

and this proves (33.02).

(34.16) can be written as

$$-\frac{r_n(u, B)k(u)}{iu} = \frac{B}{(2\pi)^{1/2}} \int_{-c}^{c} k(u - y)\phi\{B(u - y)\} h_n(y) \, dy. \tag{34.24}$$

By the definition of $k(u)$ we have

$$\begin{aligned}
\int_{-\infty}^{\infty} r_n(u, B) k(u) e^{iuy} du &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} r_n(u, B) e^{iuy} du \int_{-\infty}^{\infty} K(x) e^{-iux} dx \\
&= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} K(x) \, dx \int_{-\infty}^{\infty} r_n(u, B) e^{iu(y-x)} du \\
&= \int_{-\infty}^{\infty} K(x) R_n(y - x, B) \, dx.
\end{aligned}$$

The repeated integral here is absolutely integrable, so that inversion of the order of integration is permissible. The result can also be written as

$$\int_{-\infty}^{\infty} r_n(u, B)k(u)e^{iuy}\,du = \int_{-\infty}^{\infty} R_n(x, B)K(y - x)\,dx.$$

Integrating with respect to y, we have

$$(34.25)\qquad \int_{-\infty}^{\infty} r_n(u, B)k(u)\frac{e^{iu\alpha} - e^{ius}}{iu}\,du = \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{s}^{\alpha} K(y - x)\,dy.$$

Since $r_n(u, B)/u \in L(-\infty, \infty)$ by (34.17) and (34.18) and since $k(u)$ is bounded, it follows by the theorem of Riemann and Lebesgue that

$$\lim_{\alpha\to\infty} \int_{-\infty}^{\infty} \frac{r_n(u, B)k(u)}{iu} e^{iu\alpha}\,du = 0.$$

Letting $\alpha \to \infty$ in (34.25) we have

$$(34.26)\qquad -\int_{-\infty}^{\infty} \frac{r_n(u, B)k(u)}{iu} e^{ius}\,du = \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{s-x}^{\infty} K(y)\,dy,$$

since the repeated integral on the right is absolutely integrable.

Let

$$(34.27)\qquad \begin{aligned} J_n(s) &= \int_{-\infty}^{\infty} e^{ius}\,du \frac{B}{(2\pi)^{1/2}} \int_{-c}^{c} k(u - y)\phi\{B(u - y)\}h_n(y)\,dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ius}\,du \int_{-c}^{c} B\phi\{B(u - y)\}h_n(y)\,dy \int_{-\infty}^{\infty} K(x)e^{-ix(u-y)}\,dx. \end{aligned}$$

That this repeated integral exists and is absolutely integrable is clear if it is written as

$$J_n(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(x)\,dx \int_{-c}^{c} h_n(y)e^{iyx}\,dy \int_{-\infty}^{\infty} B\phi\{B(u - y)\}e^{iu(s-x)}\,du.$$

It follows from (34.09) that

$$(34.28)\qquad \begin{aligned} J_n(s) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} K(x)\,dx \int_{-c}^{c} h_n(y)e^{iyx}\Phi\left(\frac{s - x}{B}\right)e^{iy(s-x)}\,dy \\ &= H_n(s) \int_{-\infty}^{\infty} K(x)\Phi\left(\frac{s - x}{B}\right)dx. \end{aligned}$$

Since (34.26) is the transform of the left side of (34.24) and $J_n(s)$ is the transform of the right side, (34.26) and (34.28) are equal giving

$$(34.29)\qquad \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{s-x}^{\infty} K(y)\,dy = H_n(s) \int_{-\infty}^{\infty} K(x)\Phi\left(\frac{s - x}{B}\right)dx.$$

Since $H_n(\lambda_m) = 0$, $(m \neq n)$, (34.29) with $s = \lambda_m$ gives (33.04). Since $H_n(\lambda_n) = 1$, (34.29) becomes, with $s = \lambda_n$,

$$\int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{\lambda_n - x}^{\infty} K(y)\, dy = \int_{-\infty}^{\infty} K(x)\Phi\left(\frac{\lambda_n - x}{B}\right) dx$$

$$= \int_{-\infty}^{\infty} K(x)\left\{\Phi(0) + \int_0^{(\lambda_n - x)/B} \Phi'(y)\, dy\right\} dx.$$

Using $\Phi(0) = 1$ and (34.10), as $B \to \infty$

$$\int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{\lambda_n - x}^{\infty} K(y)\, dy = 1 + \int_{-\infty}^{\infty} K(x)\, dx \int_0^{(\lambda_n - x)/B} \Phi'(y)\, dy$$

$$= 1 + \int_{-B^{1/2}}^{B^{1/2}} K(x)\, dx \int_0^{(\lambda_n - x)/B} \Phi'(y)\, dy$$

$$+ \left(\int_{-\infty}^{-B^{1/2}} + \int_{B^{1/2}}^{\infty}\right) K(x)\left\{\Phi\left(\frac{\lambda_n - x}{B}\right) - \Phi(0)\right\} dx$$

$$= 1 + O\left(\frac{1}{B^{1/2}}\right) + o(1) = 1 + o(1).$$

But this is (33.05). This completes the proof.

CHAPTER XI

THE GENERAL UNRESTRICTED TAUBERIAN THEOREM FOR LARGER GAPS

35. Reduction of the general theorem to the basic lemma. In the last chapter we proved by an example, Theorem L, that the higher indices theorem is not true for $K(x) = e^{-x^2/2}/(2\pi)^{1/2}$. The question now arises as to whether a more stringent condition on $\{\lambda_n\}$ than the condition $\lambda_{n+1} - \lambda_n \geqq L > 0$ of the higher indices theorem would be enough to give an unrestricted Tauberian theorem for $K(x)$. (By an unrestricted Tauberian theorem we mean one where there is no restriction on the size of the coefficients of the series involved.) We shall see that such a theorem exists and that for $e^{-x^2/2}/(2\pi)^{1/2}$ the basic requirement will essentially amount to

$$\sum_1^\infty \frac{1}{\lambda_n} < \infty.$$

The following theorem is an example of the results of this chapter and is a corollary of the considerably more general Theorem LVI that follows it

THEOREM LV. *Let*

$$\sum_1^\infty a_n \int_{\lambda_n - x}^\infty K(y)\, dy = f(x)$$

where the series converges uniformly for $x \leqq X$ for all X. Let

$$\lambda_{n+1} - \lambda_n \geqq \lambda_n - \lambda_{n-1}, \qquad n > 1. \tag{35.01}$$

Let $\Lambda(u)$ be the number of $\lambda_n < u$. Let $K(x) \in L(-\infty, \infty)$ and let

$$k(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^\infty K(x) e^{-iux}\, dx.$$

Let there exist a function $k(w)$, $w = u + iv$, coinciding with $k(u)$ for real w and analytic for $v \geqq -2\delta$ for some $\delta > 0$. Let $\alpha(u)$ be a positive even function increasing for $u > 0$ such that

$$\log \left| \frac{k(w+s)}{k(w)} \right| \leqq |s|\, \alpha(|w|), \qquad v \geqq -\delta,\ |s| \leqq \delta. \tag{35.02}$$

If

$$\int_1^\infty \frac{du}{u^2} \int_1^{\alpha(u)} \frac{\Lambda(y)}{y}\, dy < \infty, \tag{35.03}$$

then

$$f(x) = O(1), \qquad x \to \infty,$$

implies

(35.04) $$a_n = O(e^{A\lambda_n}), \qquad n \to \infty,$$

for some A depending only on $K(x)$ and $\{\lambda_n\}$.

While (35.04) is not yet of the form $a_n = O(1)$, it gives a sufficient hold on a_n to make further deductions comparatively simple.[1]

Condition (35.03) states that the faster $\alpha(u)$ grows the slower $\Lambda(u)$ must grow. For $K(x) = e^{-x^2/2}/(2\pi)^{1/2}$, $k(u) = e^{-u^2/2}/(2\pi)^{1/2}$ and therefore in this case $\alpha(u)$ is essentially $|u|$. Thus (35.03) becomes

$$\int_1^\infty \frac{du}{u^2} \int_1^u \frac{\Lambda(y)}{y}\, dy < \infty.$$

This is equivalent to

$$\int_1^\infty \frac{\Lambda(y)}{y^2}\, dy < \infty$$

which in turn is equivalent to

(35.05) $$\sum_1^\infty \frac{1}{\lambda_n} < \infty.$$

It will be seen in Theorem LVIII that (35.05) is a best possible result.

To avoid the somewhat restrictive condition (35.01), the following more general theorem is necessary.

THEOREM LVI. *Let*

$$\sum_1^\infty a_n \int_{\lambda_n - x}^\infty K(y)\, dy = f(x)$$

where the series converges uniformly for $x \leqq X$ for all X. Let[2]

(35.06) $$\lambda_{n+1} - \lambda_n \geqq L > 0.$$

Let $\Lambda(u)$ be the number of $\lambda_n < u$. Let $K(x) \in L(-\infty, \infty)$ and let

$$k(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^\infty K(x) e^{-iux}\, dx.$$

Let there exist a positive even function $\alpha(u)$ increasing for $u > 0$ and let

(35.07) $$\rho_1(u) = \max_{x \geqq \alpha(u)} \int_0^\infty \frac{4x}{x^2 + y^2} \frac{\Lambda(y)}{y}\, dy,$$

[1] Here we shall use the method of this chapter to show that (35.04) leads to $a_n = O(1)$. This fact also follows from a result of Pitt. H. R. Pitt, *General Tauberian theorems*, Proceedings of the London Mathematical Society, vol. 44 (1938), p. 243, Theorem X.

[2] Condition (35.06) can be very considerably weakened without affecting the theorem and is of secondary importance here.

and for $\delta > 0$ let

$$\rho(u) = \min(\delta, \rho_1(u)). \tag{35.08}$$

Let there exist a function $k(w)$, $w = u + iv$, coinciding with $k(u)$ for real u and analytic for $v \geqq -\rho(u)$. For complex s let

$$\left|\log \frac{k(u+s)}{k(u)}\right| \leqq |s|\,\alpha(u), \qquad |s| \leqq \rho(u), \tag{35.09}$$

$$\left|\log \left|\frac{k(w+s)}{k(w)}\right|\right| \leqq M\,|w|, \qquad |s| \leqq \rho(|w|), \quad v \geqq 0, \tag{35.10}$$

for some $M > 0$. Let

$$\theta(u) = \max_{v \leqq u} \{\rho(v)\alpha(v)\}$$

and let

$$\log \left|\frac{k'(u+s)}{k(u)}\right| \leqq \theta(u), \qquad |s| \leqq \rho(u). \tag{35.11}$$

If

$$\int_1^\infty \frac{\theta(u)}{u^2}\,du < \infty, \tag{35.12}$$

then

$$f(x) = O(1)$$

as $x \to \infty$ implies

$$a_n = O(e^{A\lambda_n}), \qquad n \to \infty, \tag{35.13}$$

for some A depending only on $K(x)$ and $\{\lambda_n\}$.

This theorem is the basic result of this chapter. To complete this result the following theorem is necessary.

THEOREM LVII. *If in addition to satisfying Theorem LVI [or LV] $K(x) = O(e^{-(A+\epsilon)x})$ for some $\epsilon > 0$ as $x \to \infty$, then*

$$\limsup_{n\to\infty} |a_n| \leqq C_0 \limsup_{x\to\infty} |f(x)| \tag{35.14}$$

for some C_0 depending only on $K(x)$ and $\{\lambda_n\}$.

From (35.14), Tauberian results, such as

$$f(x) \to s$$

implies

$$\sum_1^\infty a_n = s$$

or such as

$$\liminf \{f(x) - s\} = -\Omega, \limsup \{f(x) - s\} = \Omega$$

implies

$$\limsup_{n\to\infty} \left| \sum_1^n a_n - s \right| \leqq C_1\Omega,$$

follow quite easily exactly as in the previous chapter, making it unnecessary to reconsider these results here.

That these results are best possible at least for $K(x) = e^{-x^2/2}/(2\pi)^{1/2}$ is shown by the following theorem.

THEOREM LVIII. *If* $\lambda_{n+1} - \lambda_n \geqq L > 0$[3] *and*

$$\sum_1^\infty \frac{1}{\lambda_n} = \infty, \tag{35.15}$$

then there exists a sequence $\{a_n\}$ *such that*

$$\lim_{x\to\infty} \sum_1^\infty a_n \int_{\lambda_n - x}^\infty \frac{e^{-y^2/2}}{(2\pi)^{1/2}}\, dy = s$$

but

$$\sum_1^\infty a_n$$

diverges.

We now turn to the proof of Theorem LVI. This proof resembles that of Theorem LI in many respects, although it differs radically from it in certain respects.

LEMMA 35.1. *If the conditions of Theorem LVI are satisfied, there exists a sequence of functions* $\{R_n(x)\}$ *such that*

$$\int_{-\infty}^\infty |R_n(x)|\, dx = O(e^{A\lambda_n}), \qquad n \to \infty, \tag{35.16}$$

$$R_n(x) = 0, \qquad x > x_n < \infty, \qquad n > 0, \tag{35.17}$$

$$\int_{-\infty}^\infty R_n(x)\, dx \int_{\lambda_k - x}^\infty K(x)\, dy = 0, \qquad k \neq n, \tag{35.18}$$

and

$$\int_{-\infty}^\infty R_n(x)\, dx \int_{\lambda_n - x}^\infty K(y)\, dy = 1 \tag{35.19}$$

where A depends only on $K(x)$ *and* $\{\lambda_n\}$.

[3] This condition can be very much weakened.

Before proving this lemma we shall use it to prove Theorem LVI.

Proof of Theorem LVI. Multiplying the equation

$$\sum_{k=1}^{\infty} a_k \int_{\lambda_k - x}^{\infty} K(y)\,dy = f(x)$$

by $R_n(x)$, integrating over $(-\infty, \infty)$ and inverting the order of integration and summation as is allowed since $R_n(x) = 0$ for large x, we have

$$\sum_{k=1}^{\infty} a_k \int_{-\infty}^{\infty} R_n(x)\,dx \int_{\lambda_k - x}^{\infty} K(y)\,dy = \int_{-\infty}^{\infty} f(x) R_n(x)\,dx.$$

By the biorthogonal property of $R_n(x)$ this becomes

$$a_n = \int_{-\infty}^{\infty} f(x) R_n(x)\,dx.$$

By (35.16) and $f(x) = O(1)$ this becomes

$$a_n = O(e^{A\lambda_n}), \qquad n \to \infty.$$

This proves Theorem LVI.

36. Existence of auxiliary functions. Before proceeding further we shall prove that we can assume

$$\lim_{x\to\infty} \frac{\Lambda(x)}{x} = 0. \tag{36.01}$$

For suppose that

$$\limsup_{x\to\infty} \frac{\Lambda(x)}{x} = a > 0$$

Then since $\Lambda(x)$ is non-decreasing

$$\begin{aligned} \rho_1(u) &= \max_{x \geq \alpha(u)} \int_0^{\infty} \frac{4x}{x^2+y^2} \frac{\Lambda(y)}{y}\,dy \\ &\geq \limsup_{x\to\infty} \int_x^{2x} \frac{4x}{x^2+y^2} \frac{\Lambda(y)}{y}\,dy \geq \limsup_{x\to\infty} \Lambda(x) \int_x^{2x} \frac{4x}{5x^2} \frac{dy}{2x} \\ &= \limsup_{x\to\infty} \frac{2}{5} \frac{\Lambda(x)}{x} = \frac{2}{5} a. \end{aligned} \tag{36.02}$$

But if $\rho_1(u) \geq 2a/5$, it follows from the definition of $\theta(u)$ that $\alpha(u)$ must satisfy the requirements on $\theta(u)$; that is,

$$\int_1^{\infty} \frac{\alpha(u)}{u^2}\,du < \infty. \tag{36.03}$$

Since $\rho(u) = \min(\delta, 2a/5)$ and $\alpha(u)$ satisfies the requirements of $\theta(u)$, (35.09) implies (32.07), (35.10) implies (32.08), and (35.11) implies (32.09). Thus if (36.01) is not satisfied the hypothesis of Theorem LVI implies that the hypothe-

sis of Theorem LI is fulfilled and thus in this case Theorem LVI is proved by Theorem LI. (Actually in this case Theorem LVI is somewhat more restrictive than Theorem LI.) Thus for the remainder of this chapter we shall assume that (36.01) holds. Since $\rho_1(u)$ is a decreasing function and since $\rho_1(u) \geqq 2a/5$ leads to Theorem LI, it is clear that we are interested only in the case where $\rho_1(u) \to 0$ as $u \to \infty$, and therefore where $\rho(u) \to 0$ as $u \to 0$. That $\rho(u) \to 0$ is in fact implied by (36.01).

Let $C_2, C_3, \cdots$ be positive constants which depend only on $K(x)$ and $\{\lambda_n\}$. (We shall assume that δ is fixed for any $k(w)$ and therefore also depends only on $K(x)$.) Let

$$F(z) = \prod_1^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right). \tag{36.04}$$

LEMMA 36.1. *If the conditions of Theorem LVI are satisfied, there exists a sequence of functions $\{f_n(s)\}$, analytic for $|s| > 0$, such that*

$$|f_n(s)| \leqq C_2 \frac{e^{6\theta(u)}}{|F'(\lambda_n)|}, \qquad |s| \geqq \rho(u), \tag{36.05}$$

and such that if C is a path about $s = 0$

$$\frac{1}{2\pi i}\int_C f_n(s)e^{isz} = F_n(z) \tag{36.06}$$

where

$$F_n(z) = \frac{F(z)}{F'(\lambda_n)(z - \lambda_n)}. \tag{36.07}$$

Proof of Lemma 36.1. $F_n(z)$ is defined as in (36.07). For $s \neq 0$ we define $f_n(s)$ by

$$f_n(s) = \int_0^\infty F_n(z)e^{-isz}\,dz \tag{36.08}$$

where the path of integration is along the line am $z = -$ am $s - \frac{1}{2}\pi$, and therefore along the path of integration $e^{isz} = e^{|sz|}$. Since we assume that $\Lambda(u)/u \to 0$, by Theorem XXX, $F(z) = O(e^{\epsilon|z|})$ for any ϵ. Thus by the Cauchy integral theorem we can rotate the path of integration of $f_n(s)$ through any angle less than 90°. Therefore $f_n(s)$ can be defined in any sector of the s plane whose angle is less than 180° by integrating along a fixed line in the plane (fixed for any particular sector). Thus $f_n(s)$ is analytic in such a sector excluding $s = 0$, and since the s plane can be covered by three such overlapping sectors, $f_n(s)$ is analytic in the entire s plane except for $s = 0$.

From the definition of $f_n(s)$,

$$|f_n(s)| \leqq \int_0^\infty |F_n(z)|e^{-|sz|}\,|dz|. \tag{36.09}$$

Using the prime on the product sign to denote the omission of the term $k = n$,

$$(36.10)\qquad |F_n(z)| \leq \frac{1}{|F'(\lambda_n)|}\left|\frac{1 - z^2/\lambda_n^2}{z - \lambda_n}\right| \prod_{k=1}^{\infty}{}' \left(1 + \frac{|z|^2}{\lambda_k^2}\right),$$

or

$$(36.11)\qquad |F_n(z)| \leq \frac{\lambda_n + |z|}{\lambda_n^2 |F'(\lambda_n)|} \prod_1^{\infty} \left(1 + \frac{|z|^2}{\lambda_k^2}\right).$$

Thus (36.09) gives

$$(36.12)\qquad |f_n(s)| \leq \frac{1}{\lambda_n^2 |F'(\lambda_n)|} \int_0^{\infty} (\lambda_n + x) \prod_1^{\infty} \left(1 + \frac{x^2}{\lambda_k^2}\right) e^{-x|s|}\, dx.$$

Since

$$\log \prod_1^{\infty} \left(1 + \frac{x^2}{\lambda_k^2}\right) = \int_0^{\infty} \log\left(1 + \frac{x^2}{y^2}\right) d\Lambda(y) = \int_0^{\infty} \frac{2x^2}{x^2 + y^2} \frac{\Lambda(y)}{y}\, dy,$$

(36.12) gives

$$(36.13)\quad |f_n(s)| \leq \frac{1}{\lambda_n^2 |F'(\lambda_n)|} \int_0^{\infty} (x + \lambda_n) \exp\left\{-x|s| + \int_0^{\infty} \frac{2x^2}{x^2 + y^2} \frac{\Lambda(y)}{y}\, dy\right\} dx.$$

Let

$$(36.14)\qquad J_1 = \int_0^{\alpha(u)} (x + \lambda_n) \exp\left\{-x|s| + \int_0^{\infty} \frac{2x^2}{x^2 + y^2} \frac{\Lambda(y)}{y}\, dy\right\} dx.$$

Since

$$(36.15)\qquad \frac{d}{dx} \int_0^{\infty} \frac{2x^2}{x^2 + y^2} \frac{\Lambda(y)}{y}\, dy = \int_0^{\infty} \frac{4xy^2}{(x^2 + y^2)^2} \frac{\Lambda(y)}{y}\, dy > 0,$$

it follows that

$$\begin{aligned}
\max_{x \leq \alpha(u)} \int_0^{\infty} \frac{2x^2}{x^2 + y^2} \frac{\Lambda(y)}{y}\, dy &= \int_0^{\infty} \frac{2\alpha^2(u)}{\alpha^2(u) + y^2} \frac{\Lambda(y)}{y}\, dy \\
&= \frac{1}{2} \alpha(u) \left[\int_0^{\infty} \frac{4x}{x^2 + y^2} \frac{\Lambda(y)}{y}\, dy\right]_{x=\alpha(u)} \\
&\leq \frac{1}{2} \alpha(u) \max_{x \geq \alpha(u)} \int_0^{\infty} \frac{4x}{x^2 + y^2} \frac{\Lambda(y)}{y}\, dy = \frac{1}{2} \alpha(u) \rho_1(u).
\end{aligned}$$

Since $\rho_1(u) \to 0$, $\rho(u) = \rho_1(u)$ for large u. Using the above inequality in (36.14), we have therefore for large u

$$J_1 \leq \int_0^{\alpha(u)} (x + \lambda_n) e^{-x|s| + \alpha(u)\rho(u)/2}\, dx.$$

Thus

$$(36.16)\qquad J_1 \leqq \int_0^{\alpha(u)} (x + \lambda_n)e^{\alpha(u)\rho(u)/2}\,dx < \alpha(u)(\lambda_n + \alpha(u))e^{\alpha(u)\rho(u)/2}.$$

Let

$$(36.17)\qquad J_2 = \int_{\alpha(u)}^{\infty} (x + \lambda_n) \exp\left\{-x|s| + x\int_0^{\infty} \frac{2x}{x^2 + y^2}\frac{\Lambda(y)}{y}\,dy\right\}dx.$$

Since for large u

$$\max_{x \geqq \alpha(u)} \int_0^{\infty} \frac{2x}{x^2 + y^2}\frac{\Lambda(y)}{y}\,dy = \frac{1}{2}\rho(u), \qquad J_2 \leqq \int_{\alpha(u)}^{\infty} (x + \lambda_n)e^{-x|s| + x\rho(u)/2}\,dx.$$

For $|s| \geqq \rho(u)$,

$$J_2 \leqq \int_{\alpha(u)}^{\infty} (x + \lambda_n)e^{-x\rho(u)/2}\,dx,$$

and evaluating the integral on the right we obtain

$$(36.18)\qquad J_2 \leqq \frac{e^{-\rho(u)\alpha(u)/2}}{\alpha^2(u)\rho^2(u)}\,\alpha^2(u)\{2\alpha(u)\rho(u) + 4 + 2\lambda_n\rho(u)\}$$

From the definition of $\rho(u)$ it follows for large u that

$$(36.19)\qquad \alpha(u)\rho(u) > \int_0^{\infty} \frac{\alpha^2(u)}{\alpha^2(u) + y^2}\frac{\Lambda(y)}{y}\,dy.$$

Since the integral on the right, by (36.15), is an increasing function of $\alpha(u)$ and therefore of u, it follows that for large u

$$\alpha(u)\rho(u) \geqq C_3\,.$$

Thus (36.18) becomes for large u

$$J_2 < C_4\alpha^2(u)e^{-\alpha(u)\rho(u)/2}\{\alpha(u)\rho(u) + \lambda_n\rho(u)\}.$$

Since $\alpha(u)$ is increasing and since with no restriction we can assume that $\lambda_1 > 1$,

$$(36.20)\qquad J_2 < C_5\alpha^2(u)\lambda_n\{e^{-\rho(u)\alpha(u)/2}\rho(u)\alpha(u)\} < C_6\alpha^2(u)\lambda_n\,.$$

Using (36.13) and the definition of J_1 and J_2,

$$|f_n(s)| \leqq \frac{1}{\lambda_n^2|F'(\lambda_n)|}(J_1 + J_2).$$

By (36.16) and (36.20),

$$(36.21)\qquad \begin{aligned} |f_n(s)| &\leqq \frac{1}{|F'(\lambda_n)|}\left[\alpha(u)\frac{\lambda_n + \alpha(u)}{\lambda_n^2}e^{\alpha(u)\rho(u)/2} + \frac{C_6}{\lambda_n}\alpha^2(u)\right] \\ &\leqq \frac{C_7\alpha^2(u)}{|F'(\lambda_n)|}e^{\alpha(u)\rho(u)/2}, \qquad |s| \geqq \rho(u). \end{aligned}$$

Using (36.19)

$$\alpha(u)\rho(u) > \int_0^{\alpha(u)} \frac{\alpha^2(u)}{\alpha^2(u) + y^2} \frac{\Lambda(y)}{y} dy > \frac{1}{2} \int_0^{\alpha(u)/2} \frac{\Lambda(y)}{y} dy.$$

Since $\Lambda(y) \geqq 1$, $y > \lambda_1$,

$$\alpha(u)\rho(u) > \frac{1}{2} \int_{\lambda_1}^{\alpha(u)} \frac{dy}{y} > \frac{1}{2} \log \alpha(u) - \frac{1}{2} \log \lambda_1.$$

Or

(36.22) $$\alpha(u) < C_8 e^{2\alpha(u)\rho(u)}.$$

Thus (36.21) becomes for large u

$$| f_n(s) | \leqq \frac{C_9}{| F'(\lambda_n) |} e^{5\alpha(u)\rho(u)}, \qquad | s | \geqq \rho(u).$$

Since $\alpha(u)\rho(u) \leqq \theta(u)$, and since the term "large u" is used independently of n, this result gives (36.05) for $u \geqq u_0$ for some u_0. But since $\rho(u)$ is decreasing it follows that if (36.05) holds for $u \geqq u_0$ it also holds for $u < u_0$ by adjusting the constant C_2.

We shall now prove that (36.08) implies (36.06). This is a well-known result. Let

$$F_n(z) = \sum_{k=0}^{\infty} b_k z^k.$$

Clearly

$$b_k = \frac{1}{2\pi i} \int_C \frac{F_n(z)}{z^{k+1}} dz.$$

For large $| z |$ depending on $\epsilon > 0$,

$$| F_n(z) | \leqq e^{\epsilon |z|}.$$

If $| z | = R$ on C and R is large,

$$| b_k | \leqq \frac{e^{\epsilon R}}{R^k}.$$

If k is large we can take $R = k/\epsilon$. Then

(36.23) $$| b_k | \leqq \frac{e^k \epsilon^k}{k^k} \leqq C_{10} \frac{\epsilon^k}{k!}.$$

From

$$f_n(s) = \int_0^{\infty} F_n(z) e^{-izs} dz$$

we have, if $w = izs$,

$$f_n(s) = \frac{1}{is}\int_0^{\infty} F_n\left(\frac{w}{is}\right) e^{-w}\,dw. \tag{36.24}$$

Taking w real and using the series for $F_n(z)$,

$$f_n(s) = \frac{1}{is}\int_0^{\infty} e^{-w}\sum_{k=0}^{\infty}\frac{b_k w^k}{(is)^k}\,dw = \sum_{k=0}^{\infty}\frac{b_k}{(is)^{k+1}}\,k!.$$

The inversion of integration and summation is valid since the last sum (and therefore the preceding sum) converges absolutely by (36.23) for any $s \neq 0$. Using the last equality,

$$\begin{aligned}\frac{1}{2\pi i}\int_C f_n(s)e^{isz}\,ds &= \frac{1}{2\pi i}\int_C e^{isz}\,ds\sum_{k=0}^{\infty}\frac{b_k k!}{(is)^{k+1}}\\ &= \sum_{k=0}^{\infty} b_k k!\,\frac{1}{2\pi i}\int_C \frac{e^{isz}}{(is)^{k+1}}\,ds = \sum_{k=0}^{\infty} b_k z^k = F_n(z).\end{aligned}$$

This completes the proof of this lemma.

LEMMA 36.2. *There exists an entire function $\phi(w)$ such that*

$$|\phi(u+iv)| \leqq C_{11}\frac{e^{C_{12}|v|-10\theta(u)}}{1+u^4} \tag{36.25}$$

and

$$|\phi'(u+iv)| \leqq C_{14}\frac{e^{C_{13}|v|-10\theta(u)}}{1+u^4} \tag{36.26}$$

where $\theta(u)$ satisfies Theorem LVI. *Moreover if*

$$\Phi(x) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\phi(u)e^{iux}\,du \tag{36.27}$$

then

$$|\Phi(x)| \leqq C_{15}, \qquad |\Phi'(x)| \leqq C_{16} \tag{36.28}$$

and

$$\Phi(0) = 1. \tag{36.29}$$

Proof of Lemma 36.2. By Theorem XXVI and (20.08) it follows since

$$\int_1^{\infty}\frac{10\theta(2u)}{u^2}\,du < \infty$$

that there exists an entire function $H(w)$ such that for x_1, depending only on $\theta(u)$,

$$|H(w)| \leqq \frac{e^{|v|x_1-10\theta(2u)}}{1+u^4}. \tag{36.30}$$

Also the Fourier transform of $H(w)$, $h(x)$ is such that

$$h(0) \neq 0. \tag{36.31}$$

Let

$$\phi(w) = \frac{H(w)}{h(0)}. \tag{36.32}$$

Then if $\Phi(x)$ is the transform of $\phi(u)$,

$$\Phi(0) = 1.$$

Also by (36.30) and (36.31)

$$|\Phi(x)| \leqq \frac{1}{|h(0)|} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{du}{1 + u^4}.$$

A similar result holds for $\Phi'(x)$. Thus we have proved (36.28), and (36.29).

Since $\theta(u)$ is an increasing function of $|u|$, (36.25) follows from (36.30), (36.31) and (36.32). By the Cauchy integral theorem

$$\phi'(w) = \frac{1}{2\pi i} \int_C \frac{\phi(w + s)}{s} ds$$

where here C is a circle of unit radius about $s = 0$. By (36.30), (36.31) and (36.32) it follows that

$$|\phi'(w)| \leqq C_{17} \frac{e^{x_1|v| - 10\theta(2|u| - 2)}}{1 + u^4}.$$

(36.20) follows at once, completing the lemma.

37. Proof of the basic lemma. *Proof of Lemma* 35.1. Let

$$r_n(w, B) = \frac{-iw}{k(w)} \frac{1}{2\pi i} \int_C f_n(s) k(w - s) B\phi(Bw - Bs)\, ds \tag{37.01}$$

for $v \geqq 0$, where C is the circle with center at $s = 0$ and of radius $\rho(|w|)$. The functions on the right are defined in Theorem LVI or in the preceding lemmas and are analytic for $|s| = \rho(|w|)$, $v \geqq 0$, thus making $r_n(w, B)$ analytic for $v \geqq 0$. Using

$$\log \left| \frac{k(u + s)}{k(u)} \right| \leqq |s| \alpha(u) \leqq \theta(u), \qquad |s| = \rho(u),$$

and

$$|f_n(s)| \leqq C_2 \frac{e^{6\theta(u)}}{|F'(\lambda_n)|}, \qquad |s| \geqq \rho(u),$$

gives

$$|r_n(u, B)| \leqq C_{18} \frac{|u| e^{7\theta(u)}}{|F'(\lambda_n)|} \int_0^{2\pi} B |\phi(Bu - B\rho e^{i\eta})|\, d\eta. \tag{37.02}$$

If $|u| \leqq 2\rho(0)$ this becomes using (36.25)

(37.03) $$| r_n(u, B) | \leqq \frac{C_{19}|u|}{F'(\lambda_n)} B e^{C_{20}B}, \qquad |u| \leqq 2\rho(0).$$

If $|u| > 2\rho(0)$, (37.02) becomes, using (36.25) and recalling that $\rho(|w|)$ is decreasing,

$$| r_n(u, B) | \leqq C_{21} \frac{|u| e^{7\theta(u)}}{|F'(\lambda_n)|} \frac{B e^{-10\theta(Bu/2)+C_{13}\rho(0)B}}{1 + (\tfrac{1}{2}Bu)^4}.$$

For $B > 2$ this becomes

$$| r_n(u, B) | \leqq C_{21} \frac{|u| e^{-3\theta(u)}}{|F'(\lambda_n)|} \frac{e^{C_{22}B}}{1 + u^4}, \qquad |u| > 2\rho(0).$$

Using this and (37.03)

(37.04) $$| r_n(u, B) | \leqq \frac{e^{C_{23}B}}{|F'(\lambda_n)|} \frac{|u|}{1 + u^4}.$$

From (37.01) it follows that

$$\frac{d}{du} r_n(u, B) = \frac{-i}{k(u)}\left(1 - \frac{uk'(u)}{k(u)}\right) \frac{1}{2\pi i} \int_C f_n(s) k(u - s) \phi(Bu - Bs)\, ds$$
$$- \frac{iu}{2\pi i} \int_C f_n(s) \frac{k'(u - s)}{k(u)} B\phi(Bu - Bs)\, ds$$
$$- \frac{iu}{2\pi i k(u)} \int_C f_n(s) k(u - s) B^2 \phi'(Bu - Bs)\, ds.$$

By (35.11) and (36.26) in addition to the results used above in obtaining (37.04) it follows in the same way as did (37.04) that

(37.05) $$\left| \frac{d}{du} r_n(u, B) \right| \leqq \frac{e^{C_{24}B}}{|F'(\lambda_n)|(1 + u^2)}.$$

If $R_n(x, B)$ is the Fourier transform of $r_n(u, B)$, then by (37.04)

$$| R_n(x, B) | \leqq 10 \frac{e^{C_{23}B}}{|F'(\lambda_n)|}.$$

By (37.05) and Theorem E on Fourier transforms

$$\int_{-\infty}^{\infty} x^2 | R_n(x, B) |^2 dx \leqq 10 \frac{e^{2C_{24}B}}{|F'(\lambda_n)|^2}.$$

Using these last two inequalities and the Schwarz inequality

(37.06) $$\int_{-\infty}^{\infty} | R_n(x, B) |\, dx < \frac{e^{C_{25}B}}{|F'(\lambda_n)|}.$$

Returning to the definition of $r_n(w, B)$ and again using the inequalities for $f_n(s)$, $\phi(w)$, and also the fact that

$$\log \left| \frac{k(w + s)}{k(w)} \right| \leqq M|w|, \qquad |s| \leqq \rho(|w|),$$

gives at once

$$|r_n(w, B)| \leqq \frac{e^{M|w| + 6\theta(|w|) + C_{26}B|w|}}{|F'(\lambda_n)|}. \tag{37.07}$$

For large values of u, $\theta(u) < u$, for if this were not the case, $\theta(u_n) \geqq u_n$ for a sequence $\{u_n\}$, $u_n \to \infty$. Since $u_n \to \infty$, we can assure $u_{n+1} > 2u_n$ by deleting terms. Since $\theta(u)$ is increasing

$$\int_1^\infty \frac{\theta(u)}{u^2}\, du > \sum_1^\infty \int_{u_n}^{2u_n} \frac{u_n}{u^2}\, du \geqq \sum_1^\infty \frac{u_n}{2u_n} = \infty,$$

which is impossible. Thus $\theta(u) < u$ and (37.07) becomes

$$|r_n(w, B)| \leqq \frac{e^{C_{27}B|w|}}{|F'(\lambda_n)|}. \tag{37.08}$$

It follows from (37.04) and (37.08) and Theorem C′ of Phragmén-Lindelöf that

$$|e^{iC_{27}Bw} r_n(w, B)| \leqq \frac{C_{28}}{|F'(\lambda_n)|(1 + |w|^2)}, \qquad v \geqq 0. \tag{37.09}$$

Since

$$R_n(x, B) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^\infty r_n(w, B) e^{iux}\, dx,$$

we can close the path of integration in the upper half-plane if $x > C_{27}B$. Thus

$$R_n(x, B) = 0, \qquad x > C_{27}B. \tag{37.10}$$

From the definition of $r_n(u, B)$,

$$-\frac{r_n(u, B)k(u)}{iu} = \frac{1}{2\pi i} \int_C f_n(s) k(w - s) B\phi(Bw - Bs)\, ds. \tag{37.11}$$

Exactly in the same way as in (34.26),

$$-\int_{-\infty}^\infty \frac{r_n(u, B)k(u)}{iu} e^{iu\xi}\, du = \int_{-\infty}^\infty R_n(x, B)\, dx \int_{\xi - x}^\infty K(y)\, dy. \tag{37.12}$$

Let

$$\begin{aligned} T_n(\xi) &= \int_{-\infty}^\infty e^{iw\xi}\, dw \frac{1}{2\pi i} \int_C f_n(s) k(w - s) B\phi(Bw - Bs)\, ds \\ &= \lim_{A \to \infty} \int_{-A}^A e^{iw\xi}\, dw \frac{1}{2\pi i} \int_C f_n(s) k(w - s) B\phi(Bw - Bs)\, ds. \end{aligned} \tag{37.13}$$

If we take C as a circle of radius $\rho(A)$, then inverting the order of integration, we have

$$T_n(\xi) = \lim_{A\to\infty} \frac{1}{2\pi i} \int_C f_n(s)\,ds \int_{-A}^{A} e^{iw\xi} k(w-s) B\phi(Bw - Bs)\,dw.$$

Using the Cauchy integral theorem this becomes

$$T_n(\xi) = \lim_{A\to\infty} \frac{1}{2\pi i} \int_C f_n(s)\,ds \left(\int_{-A+s}^{A+s} + \int_{-A}^{-A+s} + \int_{A+s}^{A} \right) e^{iw\xi} k(w-s) B\phi(Bw - Bs)\,dw.$$

Making an obvious change of variables,

$$T_n(\xi) = \lim_{A\to\infty} \frac{1}{2\pi i} \int_C f_n(s)\,ds \int_{-A}^{A} e^{i\xi(u+s)} k(u) B\phi(Bu)\,du$$
$$+ \lim_{A\to\infty} \frac{1}{2\pi i} \int_C f_n(s)\,ds \left(\int_{-A}^{-A+s} + \int_{A+s}^{A} \right) e^{iw\xi} k(w-s) B\phi(Bw - Bs)\,dw.$$

Inverting the order of integration in the first integral above and using

$$\frac{1}{2\pi i} \int_C f_n(s) e^{is\xi}\,ds = F_n(\xi)$$

gives

$$T_n(\xi) = F_n(\xi) \int_{-\infty}^{\infty} e^{i\xi u} k(u) B\phi(Bu)\,du + I_1 + I_2 \tag{37.14}$$

where

$$I_1 = \frac{1}{2\pi i} \int_C f_n(s)\,ds \int_{-A}^{-A+s} e^{iw\xi} k(w-s) B\phi(Bw - Bs)\,dw$$

and I_2 is the other similar term. Since C is of radius $\rho(A)$ on the path of integration $(-A, -A+s)$,

$$\left| \frac{k(w-s)}{k(-A)} \right| \leqq e^{\alpha(A)\rho(A)} \leqq e^{\theta(A)}.$$

Using this and the inequalities for $f_n(s)$ and $\phi(w)$,

$$|I_1| \leqq C_{29} \frac{e^{6\theta(A)} e^{\theta(A)} Be^{-10\theta(BA/2)+C_{12}B\rho(A)}}{|F'(\lambda_n)|\,B^4 A^4}.$$

Since $B > 2$ and since $\rho(A) \to 0$ as $A \to \infty$, it follows that

$$\lim_{A\to\infty} I_1 = 0.$$

Similarly with I_2. Thus (37.14) becomes

$$
\begin{aligned}
T_n(\xi) &= F_n(\xi) \int_{-\infty}^{\infty} e^{i\xi u} k(u) B\phi(Bu)\, du \\
&= F_n(\xi) \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\xi u} B\phi(Bu)\, du \int_{-\infty}^{\infty} K(y) e^{-iuy}\, dy \\
&= F_n(\xi) \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} K(y)\, dy \int_{-\infty}^{\infty} e^{iu(\xi - y)} B\phi(Bu)\, du \\
&= F_n(\xi) \int_{-\infty}^{\infty} K(y) \Phi\left(\frac{\xi - y}{B}\right) dy.
\end{aligned}
$$

The last result with (37.11), (37.12) and (37.13) gives

$$
(37.15) \qquad \int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{\xi - x}^{\infty} K(y)\, dy = F_n(\xi) \int_{-\infty}^{\infty} K(y) \Phi\left(\frac{\xi - y}{B}\right) dy.
$$

But $F_n(\lambda_k) = 0,\ k \neq n$. Thus

$$
(37.16) \qquad \int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{\lambda_k - x}^{\infty} K(y)\, dy = 0, \qquad k \neq n.
$$

Since $F_n(\lambda_n) = 1$, (37.15) gives

$$
\int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{\lambda_n - x}^{\infty} K(y)\, dy = \int_{-\infty}^{\infty} K(y) \left\{ \Phi(0) + \int_0^{(\lambda_n - y)/B} \Phi'(x)\, dx \right\} dy.
$$

Since $\Phi(0) = 1$, this becomes

$$
(37.17) \qquad \int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{\lambda_n - x}^{\infty} K(y)\, dy = 1 + \int_{-\infty}^{\infty} K(y)\, dy \int_0^{(\lambda_n - y)/B} \Phi'(x)\, dx.
$$

Since $| \Phi'(x) | < C_{16}$,

$$
(37.18) \qquad \left| \int_0^{(\lambda_n - y)/B} \Phi'(x)\, dx \right| \leqq C_{16} \frac{| \lambda_n - y |}{B}.
$$

Also since $| \Phi(x) | < C_{15}$

$$
(37.19) \qquad \left| \int_0^{(\lambda_n - y)/B} \Phi'(x)\, dx \right| = \left| \Phi\left(\frac{\lambda_n - y}{B}\right) - \Phi(0) \right| < 2C_{15}.
$$

Thus if C_{30} is so chosen that

$$
\left(\int_{-\infty}^{-C_{30}} + \int_{C_{30}}^{\infty} \right) | K(y) |\, dy < \frac{1}{8C_{15}},
$$

then by (37.18) and (37.19)

$$
\left| \int_{-\infty}^{\infty} K(y)\, dy \int_0^{(\lambda_n - y)/B} \Phi'(x)\, dx \right| \leqq C_{16} \frac{C_{30} + \lambda_n}{B} \int_{-C_{30}}^{C_{30}} |K(y)|\, dy + \tfrac{1}{4} < C_{31} \frac{1 + \lambda_n}{B} + \tfrac{1}{4}.
$$

If we take $B = B_n = 4C_{31}(\lambda_n + 1)$, then by (37.17) and the above inequality

$$\left| \int_{-\infty}^{\infty} R_n(x, B_n)\, dx \int_{\lambda_n - x}^{\infty} K(y)\, dy \right| \geqq 1 - \tfrac{1}{4} - \tfrac{1}{4} = \tfrac{1}{2}.$$

Thus if we set

$$\gamma_n = \int_{-\infty}^{\infty} R_n(x, B_n)\, dx \int_{\lambda_n - x}^{\infty} K(y)\, dy, \tag{37.20}$$

then $|\gamma_n| \geqq \frac{1}{2}$. Let

$$R_n(x) = \frac{R_n(x, B_n)}{\gamma_n}. \tag{37.21}$$

Then by (37.06)

$$\int_{-\infty}^{\infty} |R_n(x)|\, dx < C_{32} \frac{e^{C_{31}\lambda_n}}{|F'(\lambda_n)|}.$$

Since by Theorem XXX, $|F'(\lambda_n)| > e^{\epsilon\lambda_n}$, (35.16) follows at once. (Since $|F'(\lambda_n)| > e^{-10\lambda_n}$ would do just as well, it is clear that $\lambda_{n+1} - \lambda_n \geqq L > 0$ can be very much relaxed.) By (37.10), (35.17) follows. By (37.16), (35.18) follows. By (37.20) and (37.21), (35.19) follows. This completes the proof of the lemma and therefore of Theorem LVI.

38. Proofs of the remaining theorems. Theorem LV is proved by showing that its hypothesis implies Theorem LVI is satisfied.

Proof of Theorem LV. (35.06) and (35.09) are obviously satisfied. Clearly

$$\frac{n}{\lambda_n} - \frac{n+1}{\lambda_{n+1}} = \frac{n(\lambda_{n+1} - \lambda_n) - \lambda_n}{\lambda_n \lambda_{n+1}}.$$

Since $\lambda_{n+1} - \lambda_n$ is increasing,

$$n(\lambda_{n+1} - \lambda_n) \geqq (\lambda_{n+1} - \lambda_n) + (\lambda_n - \lambda_{n-1}) + \cdots + (\lambda_2 - \lambda_1).$$

Thus

$$\frac{n}{\lambda_n} - \frac{n+1}{\lambda_{n+1}} \geqq \frac{\lambda_{n+1} - \lambda_n - \lambda_1}{\lambda_n \lambda_{n+1}}.$$

But $\lambda_{n+1} - \lambda_n \to \infty$, otherwise we have Theorem LI. Thus it follows that for large n

$$\frac{n}{\lambda_n} > \frac{n+1}{\lambda_{n+1}}. \tag{38.01}$$

If $b > a$ and $\lambda_N \leqq b < \lambda_{N+1}$, $\lambda_n \leqq a < \lambda_{n+1}$, then clearly

$$\frac{\Lambda(a) + 1}{a} \geqq \frac{n+1}{\lambda_{n+1}}, \qquad \frac{\Lambda(b)}{b} \leqq \frac{N}{\lambda_N}.$$

If $N > n$ then by (38.01) it follows that

$$\frac{n+1}{\lambda_{n+1}} > \frac{N}{\lambda_N}$$

giving

$$\frac{\Lambda(a)+1}{a} \geqq \frac{\Lambda(b)}{b}, \qquad b > a. \tag{38.02}$$

If $n = N$ then $\Lambda(a) = \Lambda(b) = n$ and since $b > a$, (38.02) is true here also.

By (35.07)

$$\rho_1(u) = \max_{x \geq \alpha(u)} \int_0^\infty \frac{4x}{x^2+y^2} \frac{\Lambda(y)}{y}\, dy.$$

Using (38.02)

$$\rho_1(u) \leqq \max_{x > \alpha(u)} \left[\int_0^x \frac{4x}{x^2+y^2} \frac{\Lambda(y)}{y}\, dy + \frac{\Lambda(x)+1}{x} \int_x^\infty \frac{4x}{x^2+y^2}\, dy\right]$$

$$\leqq \max_{x > \alpha(u)} \left[\frac{4}{x} \int_0^x \frac{\Lambda(y)}{y}\, dy + 2\pi \frac{\Lambda(x)+1}{x}\right].$$

By (38.02) for large x

$$\frac{\Lambda(x)+1}{x} \leqq \frac{1}{x-\lambda_1} \int_{\lambda_1}^x \frac{\Lambda(y)+2}{y}\, dy < \frac{4}{x} \int_{\lambda_1}^x \frac{\Lambda(y)}{y}\, dy.$$

Thus

$$\rho_1(u) \leqq \max_{x > \alpha(u)} \left(\frac{80}{x} \int_0^x \frac{\Lambda(y)}{y}\, dy\right). \tag{38.03}$$

Clearly

$$\frac{d}{dx}\left(\frac{1}{x} \int_0^x \frac{\Lambda(y)}{y}\, dy\right) = -\frac{1}{x^2} \int_0^x \frac{\Lambda(y)}{y}\, dy + \frac{\Lambda(x)}{x^2} = -\frac{1}{x^2} \int_0^x \left(\frac{\Lambda(y)}{y} - \frac{\Lambda(x)}{x}\right) dy.$$

Using (38.02) for large a,

$$\frac{d}{dx}\left(\frac{1}{x} \int_0^x \frac{\Lambda(y)}{y}\, dy\right) \leqq \frac{1}{x^2} \int_{\lambda_1}^x \frac{dy}{y} + \frac{C_{34}}{x^2} \leqq C_{35} \frac{\log x}{x^2}.$$

Thus (38.03) gives

$$\rho_1(u) \leqq \frac{8}{\alpha(u)} \int_0^{\alpha(u)} \frac{\Lambda(y)}{y}\, dy + 8C_{35} \int_{\alpha(u)}^\infty \frac{\log x}{x^2}\, dx$$

$$\leqq \frac{8}{\alpha(u)} \int_0^{\alpha(u)} \frac{\Lambda(y)}{y}\, dy + C_{36} \frac{\log \alpha(u)}{\alpha(u)}.$$

Since $\Lambda(y) \geqq 1$, $(y > \lambda_1)$, the above inequality implies

$$\rho_1(u) \leqq \frac{C_{37}}{\alpha(u)} \int_0^{\alpha(u)} \frac{\Lambda(y)}{y}\, dy. \tag{38.04}$$

Since for large u, $\rho_1(u) \to 0$, it follows from (38.04) for large u that

(38.05) $$\theta(u) = \max_{v \leqq u} (\alpha(v)\rho(v)) \leqq C_{37} \int_0^{\alpha(u)} \frac{\Lambda(y)}{y}\, dy.$$

Thus (35.03) implies (35.12).

In the paragraph following (37.07) it was shown that for large u, $\theta(u) < u$. Thus $\alpha(u)\rho(u) < u$ for large u and therefore (35.02) implies (35.10).

Since $k(w + s)$ is analytic for $|s| \leqq \delta$ and $v \geqq -\delta$,

$$k(w + s) = k(w) + sk'(w) + O(|s|^2)$$

for small $|s|$. Thus

$$\frac{k(w + s)}{k(w)} = 1 + s\frac{k'(w)}{k(w)} + O(|s|^2).$$

From this for small $|s|$

$$\log \frac{k(w + s)}{k(w)} = s\frac{k'(w)}{k(w)} + O(|s|^2).$$

Taking am $s = -$ am $(k'(w)/k(w))$ and taking the real part of the above equation,

$$\log \left|\frac{k(w + s)}{k(w)}\right| = |s|\left|\frac{k'(w)}{k(w)}\right| + O(|s|^2).$$

Since it is given that

$$\log \left|\frac{k(w + s)}{k(w)}\right| \leqq |s|\, \alpha(|w|),$$

it follows that

$$\left|\frac{k'(w)}{k(w)}\right| \leqq \alpha(|w|), \qquad v \geqq -\delta.$$

Setting $w = u + s$, $|s| \leqq \delta$, this becomes

$$\left|\frac{k'(u + s)}{k(u + s)}\right| \leqq \alpha(|u| + |s|), \qquad |s| \leqq \delta.$$

Combining this with (35.02) it follows that

$$\log \left|\frac{k'(u + s)}{k(u + s)}\right| + \log \left|\frac{k(u + s)}{k(u)}\right| \leqq |s|\,\alpha(u) + \log \alpha(|u| + |s|).$$

Or

$$\log \left|\frac{k'(u + s)}{k(u)}\right| \leqq |s|\,\alpha(u) + \log \alpha(|u| + |s|).$$

By (36.22) $\log \alpha(u) < 2\theta(u) + \log C_8$. Thus

$$\log\left|\frac{k'(u+s)}{k(u)}\right| \leqq \theta(u) + 2\theta(2u) + \log C_8, \qquad |s| \leqq \rho(u). \tag{38.06}$$

This result while it is not exactly (35.11), serves quite as well as (35.11) in the actual proof of Theorem LVI if we replace $\theta(u)$ by the larger function $\theta(2u)$ throughout the proof. This completes the proof of Theorem LV.

In proving Theorem LVII we require the following lemma.

LEMMA 38.1. *If Theorem* LVI *is satisfied, then for any $c > 0$ and any $\epsilon > 0$*

$$\frac{k(w+\xi)}{k(w)} = O(e^{e^{\epsilon|w|}}), \qquad |\xi| \leqq c, v \geqq 0, \tag{38.07}$$

as $|w| \to \infty$.

Proof of Lemma 38.1. From (36.22)

$$\log \alpha(u) \leqq 2\theta(u) + \log C_8.$$

Thus by (35.12)

$$\int_1^\infty \frac{\log \alpha(y)}{y^2}\, dy < \infty.$$

Therefore

$$\lim_{u\to\infty} \int_u^{2u} \frac{\log \alpha(y)}{y^2}\, dy = 0.$$

Since $\alpha(u)$ is increasing this gives

$$\lim_{u\to\infty} \alpha(u) \int_u^{2u} \frac{dy}{y^2} = 0$$

or

$$\lim_{u\to\infty} \frac{\alpha(u)}{u} = 0.$$

Thus for any $\epsilon > 0$,

$$\alpha(u) < e^{\epsilon|u|}, \qquad |u| \to \infty. \tag{38.08}$$

But from (36.22) for large u

$$\frac{\log \alpha(u)}{3\alpha(u)} < \rho(u).$$

Since $\alpha(u)$ is increasing it follows from this and (38.08) for large $|u|$ that

$$\rho(u) > e^{-\epsilon|u|}. \tag{38.09}$$

By (35.10) for real ξ

$$\log\left|\frac{k(w+\xi)}{k(w)}\right| \leqq M|w|, \qquad v \geqq 0, |\xi| \leqq \rho(|w|).$$

For large $|w|$, $\rho(|w|)$ is decreasing. Thus

$$\rho(w+\xi) \geqq \rho(2|w|), \qquad |\xi| \leqq c.$$

Let $|\xi_0| \leqq c$,

$$\xi_n = \xi_0 - n\rho(2|w|),$$

and let N be the integer for which

$$-\rho(2|w|) \leqq \xi_0 - N\rho(2|w|) < \rho(2|w|).$$

Then

$$\log\left|\frac{k(w+\xi_{n-1})}{k(w+\xi_n)}\right| \leqq M|w|, \qquad 1 \leqq n \leqq N.$$

Adding from $n = 1$ to $n = N$

$$\log\left|\frac{k(w+\xi_0)}{k(w+\xi_N)}\right| \leqq NM|w|.$$

Also

$$\log\left|\frac{k(w+\xi_N)}{k(w)}\right| \leqq M|w|.$$

Adding the above two results and observing that, from the definition of N,

$$N \leqq 1 + \frac{|\xi_0|}{\rho(2|w|)} \leqq 1 + \frac{c}{\rho(2|w|)},$$

it follows that for large $|w|$

$$\log\left|\frac{k(w+\xi_0)}{k(w)}\right| \leqq \frac{2cM|w|}{\rho(2|w|)}.$$

By (38.09)

$$\log\left|\frac{k(w+\xi)}{k(w)}\right| \leqq 2cM|w|e^{2\epsilon|w|}, \qquad |\xi| \leqq c.$$

Redefining ϵ, (38.07) follows at once.

LEMMA 38.2. *If Theorem LVII is satisfied, there exists a sequence of functions $\{R_n(x, B)\}$, such that*

$$\text{(38.10)} \qquad \int_{-\infty}^{\infty} |R_n(x, B)|\,dx < C_{38},$$

$$\text{(38.11)} \qquad \int_{-\infty}^{\lambda_n/2} |R_n(x, B)|\,dx < \frac{C_{39}}{\lambda_n^{1/2}},$$

$$\text{(38.12)} \qquad R_n(x, B) = O(e^{-2Ax}), \qquad x \to \infty,$$

$$(38.13)\qquad \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_k - x}^{\infty} K(y)\,dy = 0, \qquad k \neq n,$$

and

$$(38.14)\qquad \lim_{B\to\infty} \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_n - x}^{\infty} K(y)\,dy = 1.$$

Proof of Lemma 38.2. Let

$$(38.15)\qquad r_n(u, B) = \frac{-iuBe^{-iu\lambda_n+1}}{\overline{k(u)}(2\pi)^{1/2}} \int_{u-c}^{u+c} k(y)e^{\ \cosh y/2A}\phi(By)g_n(u-y)e^{i\lambda_n y}\,dy$$

where $c = 2\pi/L$ and $\phi(u)$ and $g_n(u)$ are defined as in Lemmas 34.1 and 34.2 of Chapter X. Using (38.07) and proceeding exactly as in the proof of Lemma 33.1, (38.10) and (38.11) follow at once.

Let

$$r_n(w, B) = \frac{-iBwe}{k(w)(2\pi)^{1/2}} \int_{-c}^{c} k(w-y)e^{-\cosh(w-y)/2A}\phi(Bw - By)h_n(y)\,dy$$

for $v \geqq 0$, $w = u + iv$. Then for $w = u$ this definition coincides with (38.15). For $v \geqq 0$, $r_n(w, B)$ is analytic. Also for $v \geqq 0$, by (38.07) and (34.07),

$$(38.16)\qquad r_n(w, B) = O(B\,|\,w\,|\,e^{e^{\epsilon|w|}}e^{C_{40}B|v|}\,|\,e^{\ \cosh(u+iv)/2A}\,|)$$

as $|\,w\,| \to \infty$. Since

$$\Re \cosh \frac{u+iv}{2A} = \cosh \frac{u}{2A} \cos \frac{v}{2A},$$

if $0 \leqq v \leqq 2A$,

$$\Re \cosh \frac{u+iv}{2A} > \frac{1}{100} e^{|w|/2A},$$

and therefore

$$(38.17)\qquad e^{-\cosh w/2A} = O(e^{-(1/100)e^{|u|/2A}}).$$

Therefore if $\epsilon < \frac{1}{2}A$, (38.16) gives

$$(38.18)\qquad r_n(w, B) = O(e^{C_{41}B}e^{-e^{|w|/4A}}), \qquad 0 \leqq v \leqq 2A.$$

Since

$$R_n(x, B) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} r_n(w, B)e^{iwx}\,dw,$$

it follows from (38.18) and the Cauchy integral theorem that

$$R_n(x, B) \leqq \int_{-\infty+2iA}^{\infty+2iA} e^{\ 2Ax}\,|\,r_n(w, B)\,dw\,|,$$

and by (38.18)

$$R_n(x, B) = O(e^{-2Ax}), \qquad x \to \infty.$$

This proves (38.12).

Let

$$(38.19) \qquad T(x) = \frac{e}{2\pi} \int_{-\infty}^{\infty} e^{-\cosh w/2A} e^{iwx}\, dw.$$

By (38.17), the Cauchy integral theorem can be applied as above to (38.19) to give

$$T(x) = O(e^{-2A|x|}),$$

and thus

$$\int_{-\infty}^{\infty} |T(x)|\, dx < \infty.$$

Also by the Fourier transform theorem, Theorem E,

$$\int_{-\infty}^{\infty} T(x)\, dx = ee^{-\cosh 0} = 1.$$

If we now treat

$$(38.20) \qquad \frac{k(u) r_n(u, B)}{-iu} = \frac{Be}{(2\pi)^{1/2}} \int_{-c}^{c} k(u - y) e^{-\cosh(u-y)/2A} \phi(Bu - By) h_n(y)\, dy$$

in the same way as in the proof of Lemma 33.1 and in addition use (38.19), we have

$$(38.21) \qquad \begin{aligned} &\int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{s-x}^{\infty} K(y)\, dy \\ &\qquad = H_n(s) \int_{-\infty}^{\infty} K(x)\, dx \int_{-\infty}^{\infty} T(y) \Phi\left(\frac{s - x - y}{B}\right) dy. \end{aligned}$$

(38.13) follows at once on setting $s = \lambda_k$, $k \neq n$. If $s = \lambda_n$ then, as $B \to \infty$,

$$(38.22) \qquad \begin{aligned} \int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{\lambda_n - x}^{\infty} K(y)\, dy &= \int_{-\infty}^{\infty} K(x)\, dx \int_{-\infty}^{\infty} T(y) \Phi\left(\frac{\lambda_n - x - y}{B}\right) dy \\ &= \int_{-B^{1/2}}^{B^{1/2}} K(x)\, dx \int_{-B^{1/2}}^{B^{1/2}} T(y) \Phi\left(\frac{\lambda_n - x - y}{B}\right) dy + o(1). \end{aligned}$$

But

$$\begin{aligned} \Phi\left(\frac{\lambda_n - x - y}{B}\right) &= \Phi(0) + \int_0^{(\lambda_n - x - y)/B} \Phi'(y)\, dy \\ &= 1 + O\left(\frac{1}{B^{1/2}}\right), \qquad |x| \leq B^{1/2}, \qquad |y| \leq B^{1/2}, \qquad B \to \infty. \end{aligned}$$

Thus formula (38.22) gives (38.14). This completes the proof of the lemma.

Proof of Theorem LVII. We have

$$\sum_1^\infty a_n \int_{\lambda_n - x}^\infty K(y)\, dy = f(x) \tag{38.23}$$

with $f(x)$ bounded, and by (35.13)

$$a_n = O(e^{A\lambda_n}). \tag{38.24}$$

Multiplying (38.23) by $R_n(x, B)$

$$\int_{-\infty}^\infty R_n(x, B)\, dx \sum_1^\infty a_k \int_{\lambda_k - x}^\infty K(y)\, dy = \int_{-\infty}^\infty R_n(x, B) f(x)\, dx. \tag{38.25}$$

Clearly

$$\int_{-\infty}^\infty | R_n(x, B) |\, dx \sum_1^\infty \left| a_k \int_{\lambda_k - x}^\infty K(y)\, dy \right|$$
$$\leqq \int_{-\infty}^\infty | R_n(x, B) e^{(A+\epsilon)x} |\, dx \sum_1^\infty | a_k e^{-(A+\epsilon)\lambda_k} | e^{(A+\epsilon)(\lambda_k - x)} \int_{\lambda_k - x}^\infty | K(y) |\, dy.$$

Since $K(x) = O(e^{-(A+\epsilon)x})$

$$\int_{\lambda_k - x}^\infty | K(y) |\, dy < Ce^{-(\lambda_k - x)(A+\epsilon)}$$

for some C independent of k and x. Thus

$$\int_{-\infty}^\infty | R_n(x, B) |\, dx \sum_1^\infty \left| a_k \int_{\lambda_k - x}^\infty K(y)\, dy \right|$$
$$\leqq C \int_{-\infty}^\infty | R_n(x, B) e^{(A+\epsilon)x} |\, dx \sum_1^\infty | a_k | e^{-(A+\epsilon)\lambda_k}.$$

Thus by (38.10), (38.12), and (38.24) the order of integration and summation in the left side of (38.25) can be inverted. Using (38.13) and (38.14) this gives

$$a_n = \lim_{B \to \infty} \int_{-\infty}^\infty R_n(x, B) f(x)\, dx.$$

By (38.10) and (38.11) this becomes

$$\limsup_{n \to \infty} | a_n | \leqq C_{38} \limsup_{x \to \infty} | f(x) |,$$

which completes the proof.

Proof of Theorem LVIII. Let

$$G(w) = \prod_1^\infty \left(1 + \frac{w}{\lambda_n}\right) e^{-w/\lambda_n}.$$

Let $w = re^{i\theta}$. Then if $\Lambda(t)$ is the number of $\lambda_n < t$,

$$\log |G(re^{i\theta})| = \int_0^\infty \left\{\log\left|1 + \frac{re^{i\theta}}{t}\right| - \frac{r\cos\theta}{t}\right\} d\Lambda(t).$$

Integrating by parts,

$$\log |G(re^{i\theta})| = -r^2 \int_0^\infty \frac{t(2\cos^2\theta - 1) + r\cos\theta}{t^2 + 2rt\cos\theta + r^2} \frac{\Lambda(t)}{t^2} dt.$$

For $|\theta| \leqq \frac{1}{4}\pi$, $2\cos^2\theta - 1 \geqq 0$ and therefore

$$\log |G(re^{i\theta})| \leqq -r^2 \int_0^\infty \frac{r\cos\theta}{t^2 + 2rt\cos\theta + r^2} \frac{\Lambda(t)}{t^2} dt$$

$$\leqq -r\cos\theta \int_0^r \frac{r^2}{4r^2} \frac{\Lambda(t)}{t^2} dt$$

$$\leqq -\tfrac{1}{4} r\cos\theta \int_0^r \frac{\Lambda(t)}{t^2} dt, \qquad |\theta| \leqq \tfrac{1}{4}\pi.$$

But (35.15) is equivalent to

(38.26) $$\lim_{r\to\infty} \int_0^r \frac{\Lambda(t)}{t^2} dt = \infty.$$

Thus for any $a > 0$

(38.27) $$\log |G(re^{i\theta})| \leqq -3ar\cos\theta, \qquad |\theta| \leqq \tfrac{1}{4}\pi,$$

for sufficiently large r. By deleting some of the λ_n it is always possible to make

$$\lim_{t\to\infty} \frac{\Lambda(t)}{t} = 0$$

without affecting (38.26). Thus if

$$F(w) = \prod_1^\infty \left(1 - \frac{w^2}{\lambda_n^2}\right),$$

then Theorem XXX holds. Thus by (38.27) and (21.06) for large r

(38.28) $$\left|\frac{G(re^{i\theta})}{F(re^{i\theta})}\right| \leqq e^{-2ar\cos\theta}, \quad |\theta| \leqq \tfrac{1}{4}\pi, \ |re^{i\theta} - \lambda_n| \geqq \tfrac{1}{4}L.$$

Clearly

$$\frac{G(w)}{F(w)} = 1 \Big/ \prod_1^\infty \left(1 - \frac{w}{\lambda_n}\right) e^{w/\lambda_n}.$$

Thus as above

$$\log \left|\frac{G(re^{i\theta})}{F(re^{i\theta})}\right| = -r^2 \int_0^\infty \frac{t(1 - 2\cos^2\theta) + r\cos\theta}{t^2 - 2rt\cos\theta + r^2} \frac{\Lambda(t)}{t^2} dt.$$

For $\frac{1}{4}\pi \leqq |\theta| \leqq \frac{1}{2}\pi$, $1 - 2\cos^2\theta \geqq 0$, and thus

$$\log\left|\frac{G(re^{i\theta})}{F(re^{i\theta})}\right| \leqq -10(1-2\cos^2\theta)\int_{\lambda_{10}}^{\infty}\frac{r^2}{t^2+r^2}\frac{dt}{t} - r\cos\theta\int_0^{\infty}\frac{r^2}{t^2+r^2}\frac{\Lambda(t)}{t^2}\,dt$$

$$\leqq -5(1-2\cos^2\theta)\int_{\lambda_{10}}^{r}\frac{dt}{t} - r\cos\theta\,\frac{1}{2}\int_0^{r}\frac{\Lambda(t)}{t^2}\,dt.$$

Therefore

$$\log\left|\frac{G(re^{i\theta})}{F(re^{i\theta})}\right| \leqq -5(1-2\cos^2\theta)\log\frac{r}{\lambda_{10}} - \frac{1}{2}r\cos\theta\int_0^{r}\frac{\Lambda(t)}{t^2}\,dt. \tag{38.29}$$

From (38.29) if $w = u + iv$,

$$\left|\frac{G(iv)}{F(iv)}\right| \leqq \frac{\lambda_{10}^5}{|v|^5}. \tag{38.30}$$

By (38.26) for any a and sufficiently large r (38.29) gives

$$\log\left|\frac{G(re^{i\theta})}{F(re^{i\theta})}\right| \leqq -2\log r - ar\cos\theta, \qquad \tfrac{1}{4}\pi \leqq |\theta| \leqq \tfrac{1}{2}\pi.$$

Combining this with (38.28), if $w = u + iv$,

$$\frac{G(w)}{F(w)} = O\left(\frac{e^{-au}}{|w|^2}\right), \qquad |w| \to \infty, \quad u \geqq 0, \quad |w-\lambda_n| \geqq \tfrac{1}{4}L. \tag{38.31}$$

Let

$$f(y) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{G(w)}{F(w)}\,e^{-yw}\,dw. \tag{38.32}$$

Since (38.31) holds for any a, we can close the path of integration in the above integral to the right for any real y giving

$$f(y) = \sum_1^{\infty}\frac{e^{-\lambda_n y}G(\lambda_n)}{F'(\lambda_n)}. \tag{38.33}$$

By (38.30) it also follows from (38.32) that $f(y)$ is uniformly bounded for real y. Thus there exists

$$\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(y)e^{-y^2/2}\,dy = s. \tag{38.34}$$

By (38.27) and (21.07), (38.33) converges uniformly for $y \geqq -x$. Thus

$$\frac{1}{(2\pi)^{1/2}}\int_{-x}^{\infty} f(y)e^{-y^2/2}\,dy = \sum_1^{\infty}\frac{G(\lambda_n)}{F'(\lambda_n)}\frac{1}{(2\pi)^{1/2}}\int_{-x}^{\infty}e^{-\lambda_n y - y^2/2}\,dy.$$

Making an obvious change of variable in each integral on the right

$$\frac{1}{(2\pi)^{1/2}}\int_{-x}^{\infty} f(y)e^{-y^2/2}\,dy = \sum_1^{\infty}\frac{G(\lambda_n)e^{\lambda_n^2/2}}{F'(\lambda_n)}\int_{-x+\lambda_n}^{\infty}\frac{e^{-y^2/2}}{(2\pi)^{1/2}}\,dy.$$

By (38.34) it follows that

$$(38.35)\qquad \lim_{x\to\infty}\sum_{1}^{\infty}\frac{G(\lambda_n)e^{\lambda_n^2/2}}{F'(\lambda_n)}\int_{x+\lambda_n}^{\infty}\frac{e^{-y^2/2}}{(2\pi)^{1/2}}\,dy = s.$$

On the other hand (21.05) implies here, for any $\epsilon > 0$,

$$F'(w) = O(e^{\epsilon|w|}), \qquad |w| \to \infty.$$

Thus for large n

$$(38.36)\qquad \frac{1}{|F'(\lambda_n)|} > e^{-\lambda_n}.$$

Also for $0 \leqq x \leqq 1$,

$$\frac{d}{dx}[(1+x)e^{-x} - e^{-x^2}] = x[2e^{-x^2} - e^{-x}] > 0.$$

Since for $x = 0$

$$(1+x)e^{-x} \geqq e^{-x^2},$$

it is true for $0 \leqq x \leqq 1$. Thus for $u > 0$,

$$\begin{aligned} G(u) &\geqq \prod_{\lambda_n\leqq u} e^{-u/\lambda_n}\prod_{\lambda_n>u}\left(1+\frac{u}{\lambda_n}\right)e^{-u/\lambda_n} \\ &\geqq \prod_{\lambda_n\leqq u} e^{-u/\lambda_n}\prod_{\lambda_n>u} e^{-u^2/\lambda_n^2}. \end{aligned}$$

Since $\lambda_n \geqq nL$, it follows that

$$\sum_{\lambda_n<u}\frac{1}{\lambda_n} \leqq \sum_{nL\leqq u}\frac{1}{nL} < \frac{1}{L}\left(1+\log\frac{u}{L}\right)$$

and

$$\sum_{\lambda_n>u}\frac{1}{\lambda_n^2} < \sum_{nL>u}\frac{1}{n^2L^2} \leqq \frac{1}{L}\,\frac{1}{u-L}.$$

Thus

$$G(u) \geqq \exp\{-(1+\log u/L)u/L - u^2/L(u-L)\}.$$

And therefore for large n

$$G(\lambda_n) \geqq e^{-10\lambda_n \log \lambda_n/L}.$$

Combining this with (38.26) it follows that

$$\lim_{n\to\infty}\frac{G(\lambda_n)e^{\lambda_n^2/2}}{|F'(\lambda_n)|} = \infty.$$

But these are the coefficients of the series (38.35) and therefore their sum must diverge. This proves Theorem LVIII.

CHAPTER XII

ON RESTRICTIONS NECESSARY FOR CERTAIN HIGHER INDICES THEOREMS

39. A restricted higher indices theorem. We have in the last two chapters succeeded in obtaining general theorems which state that under certain conditions on $K(x)$ and $\{\lambda_n\}$, and with no restriction whatever on $\{a_n\}$,

$$\lim_{x\to\infty} \sum_1^\infty a_n \int_{\lambda_n - x}^\infty K(y)\,dy = s$$

implies that

$$\sum_1^\infty a_n = s.$$

In the case of $K(x) = e^{-x^2/2}/(2\pi)^{1/2}$ there was no general theorem if λ_n satisfied nothing more than $\lambda_{n+1} - \lambda_n \geqq L > 0$. In the example we gave to show that this was the case

$$a_n = \frac{(-1)^n e^{n^2/2}}{n!}. \tag{39.01}$$

Here a_n is very large. The question arises as to whether or not it is possible for

$$\lim_{x\to\infty} \sum_1^\infty a_n \int_{\lambda_n - x}^\infty \frac{e^{-y^2/2}}{(2\pi)^{1/2}}\,dy = s, \qquad \lambda_{n+1} - \lambda_n \geqq L > 0,$$

and yet for $\sum a_n$ to diverge if a_n is somewhat smaller than in (39.01). We shall show that this is not possible. In other words with a very mild restriction on the size of a_n and $\lambda_{n+1} - \lambda_n \geqq L$, a Tauberian theorem follows for $K(x) = e^{-x^2/2}/(2\pi)^{1/2}$.

A general result of this type is true for a large class of $K(x)$ which fall under the unrestricted Tauberian theorem of the last chapter. Actually we shall give a complete proof only for $K(x) = e^{-x^2/2}/(2\pi)^{1/2}$. The reason for this is that a general statement would be very complicated and lack precision. However the method of proof used will be easily applicable to other $K(x)$, and wherever any part of the proof for $e^{-x^2/2}/(2\pi)^{1/2}$ does not generalize in an obvious manner we shall point out exactly how that part does generalize.

The theorem for $K(x) = e^{-x^2/2}/(2\pi)^{1/2}$ follows.

THEOREM LIX. *Let*

$$\sum_1^\infty a_n \int_{\lambda_n - x}^\infty \frac{e^{-y^2/2}}{(2\pi)^{1/2}}\,dy = f(x). \tag{39.02}$$

If $\lambda_{n+1} - \lambda_n \geqq L > 0$ *and*

$$a_n = O\left(\exp\left\{\frac{1}{2}\lambda_n^2 - \frac{2\lambda_n}{L}\log\lambda_n - \frac{2\lambda_n}{L}\gamma(\lambda_n)\right\}\right) \tag{39.03}$$

where $\gamma(u)$ *is increasing,* $\gamma'(u) < m/u$, *and*

$$\int_1^\infty e^{-\gamma(v)}\frac{dv}{v} < \infty, \tag{39.04}$$

then

$$\lim_{x\to\infty} f(x) = s \tag{39.05}$$

implies

$$\sum_1^\infty a_n = s, \tag{39.06}$$

or

$$\liminf_{x\to\infty}\{f(x) - s\} = -\Omega, \limsup_{x\to\infty}\{f(x) - s\} = \Omega \tag{39.07}$$

implies

$$\limsup_{n\to\infty}\left|\sum_1^n a_k - s\right| \leqq C_0\Omega \tag{39.08}$$

where C_0 *depends only on* L.

How Theorem LIX would be stated for $K(x)$ other than $e^{-x^2/2}/(2\pi)^{1/2}$ will be clear from the proof.

Theorem LIX is a best possible result in the following sense.

THEOREM LX. *For any* $A > 0$ *there exists a sequence* $\{a_n\}$ *such that*

$$a_n = O(\exp\{\tfrac{1}{2}n^2L^2 - 2n\log n - An\}), \tag{39.09}$$

and

$$\lim_{x\to\infty}\sum_1^\infty a_n\int_{nL-x}^\infty \frac{e^{-y^2/2}}{(2\pi)^{1/2}}dy = s. \tag{39.10}$$

But

$$\lim_{n\to\infty}|a_n| = \infty. \tag{39.11}$$

The proof of Theorem LX is quite simple and can be given at once.

Proof of Theorem LX. Let

$$f(y) = -\frac{1}{2\pi i}\int_{-i\infty+1/2}^{i\infty+1/2}\frac{e^{-wLy-2Aw}}{\sin \pi w\Gamma(1+2w)}\frac{dw}{(1+w)^3}. \tag{39.12}$$

By Stirling's formula, (41.07), if $w = u + iv$ and $|\text{am}\, w| \leqq \frac{3}{4}\pi$, then

$$|\Gamma(1+u+iv)| \sim (2\pi)^{1/2}\exp\{-v\tan^{-1}v/u + (u+\tfrac{1}{2})\log|w| - u\}. \tag{39.13}$$

Thus

$$\frac{1}{\Gamma(2 + 2iv)} = O(e^{\pi|v|})$$

and therefore (39.12) implies that $f(y)$ is uniformly bounded $(-\infty < y < \infty)$.

Also by (39.13) the path of integration in (39.12) can be closed in the right half-plane to give

$$f(y) = \sum_1^\infty \frac{(-1)^n e^{-nLy-2An}}{\pi(2n)!(1+n)^3}.$$

Multiplying each side by $e^{-y^2/2}$ and integrating

$$\int_{-x}^\infty f(y)e^{-y^2/2}\,dy = \sum_1^\infty \frac{(-1)^n e^{-2An}}{\pi(2n)!(1+n)^3}\int_{-x}^\infty e^{-nLy-y^2/2}\,dy$$

$$= \sum_1^\infty \frac{(-1)^n e^{-2An+n^2L^2/2}}{\pi(2n)!(1+n)^3}\int_{-x}^\infty e^{-(y+nL)^2/2}\,dy.$$

Or

$$\int_{-x}^\infty e^{-y^2/2}f(y)\,dy = \sum_1^\infty \frac{(-1)^n e^{n^2L^2/2-2An}}{\pi(2n)!(1+n)^3}\int_{nL-x}^\infty e^{-y^2/2}\,dy.$$

Since $f(y)$ is uniformly bounded, this gives (39.10) at once with

$$a_n = \frac{(-1)^n e^{n^2L^2/2-2An}}{\pi(1+n)^3(2n)!}.$$

That (39.09) and (39.11) are satisfied follows at once.

40. Proof of the theorem. We now turn to the proof of Theorem LIX. As stated before, the method of proof will be a general one. In order to best show this we shall occasionally use the more general notation $K(x)$, $k(u)$ and

$$\max_{|\xi|\leqq\pi L}\left|\frac{k(w+\xi)}{k(w)}\right| \leqq e^{\alpha(|w|)}, \qquad v \geqq 0.$$

In our proof, of course $K(x) = e^{-x^2/2}/(2\pi)^{1/2}$, $k(u) = e^{-u^2/2}/(2\pi)^{1/2}$ and $\alpha(|w|) = \pi L\,|w| + \frac{1}{2}\pi^2L^2$. For the rest of the chapter $C_1, C_2, \cdots$ will be used to represent positive constants depending only on L.

LEMMA 40.1. *There exists a set of entire functions $\{H_n(s)\}$ such that*

(40.01) $$H_n(\lambda_k) = 0, \qquad (k \neq n), \qquad H_n(\lambda_n) = 1.$$

$H_n(x) \in L(-\infty, \infty)$ and its Fourier transform is $h_n(u)$ where

(40.02) $$h_n(u) = 0, \qquad |u| > \pi/L,$$

(40.03) $$|h_n(u)| < C_1\lambda_n^{C_2},$$

(40.04) $$|h_n'(u)| < C_3\lambda_n^{C_4}.$$

Lemma 40.1 would remain the same for any $K(x)$.

Proof of Lemma 40.1. n is kept fixed throughout the proof of this lemma. Let

$$\sigma_k = kL \text{ if } | kL - \lambda_m + \lambda_n | > \tfrac{1}{2}L \text{ for all } m > 0,$$

$$\sigma_k = kL \text{ if } kL - \lambda_m + \lambda_n = \tfrac{1}{2}L \text{ for some } m,$$

$$\sigma_k = \lambda_m - \lambda_n \quad \text{if } -\tfrac{1}{2}L \leqq kL - \lambda_m + \lambda_n < \tfrac{1}{2}L.$$

Then

$$| \sigma_k - kL | \leqq \tfrac{1}{2}L$$

and

$$\sigma_{k+1} - \sigma_k \geqq \tfrac{1}{2}L.$$

Let

$$T(s) = \prod_1^\infty \left(1 - \frac{s}{\sigma_k}\right)\left(1 - \frac{s}{\sigma_{-k}}\right).$$

Then as in (34.05)

$$| T(s) | \leqq C_5(1 + | s |)^{C_6} e^{\pi|t|/L} \tag{40.05}$$

where $t = \Im s$.

From the definition of σ_k it is clear that $\{\sigma_k\}$ contains $\{\lambda_m - \lambda_n\}$. Also if $| \lambda_n/L + 10 - N | < 1$ and if $k < -N$, then $\sigma_k \neq \lambda_m - \lambda_n$ for any m since for $k < -N$, σ_k lies to the left of all $\lambda_m - \lambda_n$. Thus if C_7 is an integer and $C_7 > C_6 + 4$ and if

$$H_n(s + \lambda_n) = T(s) \prod_N^{N+C_7}{}' \left(\frac{1}{1 - s/\sigma_{-k}}\right),$$

then (40.01) is satisfied. By (40.05)

$$| H_n(s) | \leqq C_5(1 + | s - \lambda_n |)^{C_6} e^{\pi|t|/L} \prod_N^{N+C_7} \frac{kL}{| kL + s - \lambda_n |}.$$

Since $| NL - \lambda_n | \leqq 11L$, if $| s | > (C_7 + 20)L$,

$$| H_n(s) | \leqq \frac{C_8(1 + | s - \lambda_n |)^{C_6} e^{\pi|t|/L}}{(1 + | s |)^{C_6+4}}. \tag{40.06}$$

Since $H_n(s)$ is an entire function this must also hold for $| s | \leqq (C_7 + 20)L$ if C_8 is adjusted.

Let

$$h_n(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} H_n(s) e^{-ius} \, ds. \tag{40.07}$$

By (40.06) if $u > \pi/L$, the path of integration in (40.07) can be closed in the lower half-plane giving $h_n(u) = 0$, $u > \pi/L$. A similar result holds for $u < -\pi/L$ giving (40.02). From (40.06) and (40.07)

$$|h_n(u)| \leqq C_8 \int_{-\infty}^{\infty} \frac{(1+|s-\lambda_n|)^{C_8}}{(1+|s|)^{C_8+4}}\, ds$$

from which (40.03) follows at once. (40.04) follows similarly. This completes the proof of the lemma.

[In a general theorem it would be necessary at this point to set up another lemma to prove the existence of a function, $\psi(w)$, analytic in the upper half-plane, $v \geqq 0$, and such that

$$|\psi(u)e^{\alpha(u)}| \leqq e^{\theta(u)}$$

where $\theta(u)$ is even and monotone increasing for $u > 0$, and

$$\int_1^{\infty} \frac{\theta(u)}{u^2}\, du < \infty.$$

Also $\psi(w)$ is to be of as small an order as possible in the upper half-plane speaking qualitatively. To be precise $\psi(w)$ is constructed as follows:

Let N be the smallest integer so that

$$\int_1^{\infty} \frac{\alpha(u)}{u^{N+2}}\, du < \infty.$$

Obviously $N > 0$ otherwise the requirements of Theorem LI would be satisfied and an unrestricted Tauberian theorem would be true. For $e^{-x^2/2}$, $N = 1$. Whatever N may be, let

$$U(R, \theta) = \frac{1}{\pi}\left(\int_{-\infty}^{-1} + \int_1^{\infty}\right)\alpha(\xi)\, \frac{R^{N+1}\xi \sin (N+1)\theta - R^{N+2}\sin N\theta}{\xi^{N+1}(R^2 - 2R\xi\cos\theta + \xi^2)}\, d\xi.$$

Then since

$$\begin{aligned}\frac{R^{N+1}\xi \sin (N+1)\theta - R^{N+2}\sin N\theta}{\xi^{N+1}(R^2 - 2R\xi\cos\theta + \xi^2)} &= \mathfrak{I}\, \frac{(Re^{i\theta})^{N+1}}{(\xi - Re^{i\theta})\xi^{N+1}} \\ &= \mathfrak{I}\left\{\frac{1}{\xi - Re^{i\theta}} - \frac{1}{\xi} - \frac{Re^{i\theta}}{\xi^2} - \cdots - \frac{(Re^{i\theta})^N}{\xi^{N+1}}\right\},\end{aligned}$$

$U(R, \theta)$ is analytic in the upper half-plane and is equal to $\alpha(R)$ for $R > 1$ and $\theta = 0$, or $\theta = \pi$. Let $V(R, \theta)$ be the conjugate function to $U(R, \theta)$. Then if $w = Re^{i\theta}$

$$\psi(w) = e^{-U(R,\theta) - iV(R,\theta)}.$$

In case $K(x) = e^{-x^2/2}/(2\pi)^{1/2}$ it will suffice to take $\psi(w) = \Gamma(1 - 2iw/L)$.]

Lemma 40.2. *There exists a sequence of functions $\{R_n(x, B)\}$ such that*

$$\int_{-\infty}^{\infty} |R_n(x, B)|\, dx < C_9 \lambda_n^{C_{10}}, \tag{40.08}$$

$$R_n(x, B) = O(\exp\{\tfrac{3}{2}Lx - \theta(ce^{Lx/2})\}), \tag{40.09}$$

if $\theta(x)$ *is monotone increasing and*

$$\int_1^\infty \frac{\theta(x)}{x^2}\,dx < \infty, \tag{40.10}$$

and c *is a positive constant depending on* B. *Also*

$$\int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_k - x}^{\infty} K(y)\,dy = 0, \qquad k \neq n, \tag{40.11}$$

$$\lim_{B\to\infty} \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_n - x}^{\infty} K(y)\,dy = 1. \tag{40.12}$$

Proof of Lemma 40.2. Let $\phi(w)$ be defined as in Lemma 36.3. Let $\psi(w) = \Gamma(1 - 2iw/L)$. Let

$$\Psi(s) = \frac{L}{2} e^{-e^{sL/2}} e^{sL/2}.$$

Then it follows easily that

$$\int_{-\infty}^{\infty} \Psi(s) e^{ius}\,ds = \Gamma\left(1 - \frac{2iu}{L}\right)$$

and in particular then that

$$\int_{-\infty}^{\infty} \Psi(s)\,ds = 1.$$

In general terms let

$$r_n(w, B) = \frac{-iwB}{k(w)(2\pi)^{1/2}} \int_{-\pi/L}^{\pi/L} h_n(\xi) k(w - \xi) \psi(w - \xi) \phi(Bw - B\xi)\,d\xi. \tag{40.13}$$

In the case under discussion here

$$r_n(w, B) = \frac{-iwB}{(2\pi)^{1/2}} \int_{-\pi/L}^{\pi/L} h_n(\xi) e^{w\xi - \xi^2/2} \Gamma\left\{1 - \frac{2i}{L}(w - \xi)\right\} \phi(Bw - B\xi)\,d\xi.$$

Since

$$\max_{|\xi| \leq \pi/l} | e^{u\xi} | = e^{\pi|u|/l}$$

and since

$$\max_{|\xi| \leq \pi/l} \left| \Gamma\left(1 - \frac{2i}{L}(u - \xi)\right) \right| = O(|u| e^{-\pi|u|/l}),$$

it follows that

$$| r_n(u, B) | \leq C_{12} B |u| (1 + |u|) \int_{-\pi/L}^{\pi/L} | h_n(\xi) \phi(Bu - B\xi) |\,d\xi.$$

Using (40.03) and Lemma 36.3, we get in the same way as in Lemma 33.1

$$| r_n(u, B) | \leqq C_{13} \frac{\lambda_n^{C_2}}{1 + u^2}.$$

Proceeding similarly we obtain a similar result for $r_n'(u, B)$, and using these results just as in Lemma 33.1 gives

$$\int_{-\infty}^{\infty} | R_n(x, B) | \, dx < C_{14} \lambda_n^{15}$$

where $R_n(x, B)$ is the Fourier transform of $r_n(u, B)$. This proves (40.08).

Exactly as (38.20) gives (38.21), (40.13) gives

$$\int_{-\infty}^{\infty} R_n(x, B) \, dx \int_{s-x}^{\infty} K(y) \, dy = H_n(s) \int_{-\infty}^{\infty} K(x) \, dx \int_{-\infty}^{\infty} \Psi(z) \Phi\left(\frac{s - x - z}{B}\right) dz.$$

(40.11) follows at once on setting $s = \lambda_k$. Setting $s = \lambda_n$ and proceeding as in (38.22) gives (40.12).

It is the condition (40.09) which causes us to write this theorem for a particular case rather than in general terms. For this condition is obtained by considering integrals along various curves in the complex plane and does not express itself at all simply in general terms except at a considerable sacrifice of precision.

Here

$$R_n(x, B) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-iwx} \, dw \left\{ - iBw \int_{-\pi/L}^{\pi/L} h_n(\xi) e^{w\xi - \xi^2/2} \cdot \Gamma\left(1 - \frac{2i}{L}(w - \xi)\right) \phi(Bw - B\xi) \, d\xi \right\}. \tag{40.14}$$

By the Cauchy integral theorem the path of integration can be deformed to P where part of P consists of a semicircle in the upper half-plane of radius r and center at $w = 0$ and the rest of P is $(-\infty, -r)$ and (r, ∞). Thus using (40.03) for $h_n(\xi)$,

$$\max_{|\xi| < \pi/L} \left| e^{w\xi} \Gamma\left(1 - \frac{2i}{L}(w - \xi)\right) \right| = O(| w | e^{Cv \log|w| / L}), \qquad v \geqq 0,$$

and Lemma 36.3 for $\phi(Bw - B\xi)$, (40.14) gives

$$| R_n(x, B) | \leqq C_{16} \lambda_n^{C_2} \frac{1}{L} \cdot \int_P e^{-ix} | w |^2 \frac{\exp\{C_{17} Bv - \theta(\tfrac{1}{2}Bu) + B(2v \log | w |)\}}{1 + B^4 u^4} | dw |. \tag{40.15}$$

Let the radius of the semicircle in P be

$$r = \exp\{\tfrac{1}{2}L(x - C_{17}B - 1)\}.$$

Clearly (40.15) can be written as

$$| R_n(x, B) | \leqq C_{16}\lambda_n^{C_2} Br^3 \int_0^\pi \exp\left\{r \sin t\left(-x + C_{17}B + \frac{2}{L}\log r\right) - \theta(\tfrac{1}{2}Br\cos t)\right\} dt$$
$$+ C_{16}\lambda_n^{C_2} B\left(\int_{-\infty}^{-r} + \int_r^\infty\right)\frac{u^2 e^{-\theta(Bu/2)}}{1 + B^4u^4}du.$$

Using the definition of r this gives

$$| R_n(x, B) | \leqq C_{16}\lambda_n^{C_2} Br^3 \int_0^\pi e^{-r\sin t - \theta((Br/2)\cos t)}\, dt + C_{18}\lambda_n^{C_2} Be^{-\theta(Br/2)}$$
$$\leqq C_{19}\lambda_n^{C_2} Br^3 \left\{\int_{\pi/4}^{3\pi/4} e^{-r\sin t}\, dt + 2\int_0^{\pi/4} e^{-\theta((Br/2)\cos t)}\, dt + e^{-\theta(Br/2)}\right\}$$
$$\leqq C_{20}\lambda_n^{C_2} Br^3 \{e^{-r/2^{1/2}} + e^{-\theta(Br/4)}\}.$$

Since $\theta(u)/u \to 0$ as $u \to \infty$, as we have shown several times, the last inequality gives for large x, $R_n(x, B) = O(Br^3 e^{-\theta(Br/4)})$. This gives (40.09) and completes the proof of the lemma.

[(40.09) depends on the $K(x)$ used and will be very different for different $K(x)$. The rest of the proof involves the use of (40.09) and therefore the detail is significant only for $e^{-x^2/2}$ although the general idea involved in proving Lemma 40.4 does not change with $K(x)$.]

LEMMA 40.3. *If $\gamma(u)$ satisfies (39.04), there exists a monotone increasing function, $\delta(u)$, such that $\delta(u) \to \infty$ as $u \to \infty$,*

(40.16) $$\delta'(u) = o(1/u),$$

and

(40.17) $$\int_1^\infty \frac{e^{-\gamma(u)+\delta(u)}}{u}\, du < \infty.$$

Proof of Lemma 40.3. Let

$$\epsilon_n = \int_{2^n}^\infty e^{-\gamma(y)} \frac{dy}{y}.$$

Then $\{\epsilon_n\}$ is a monotone decreasing sequence and $\epsilon_n \to 0$ as $n \to \infty$. Let $n_1, n_2, \cdots$ be a sequence of increasing integers such that

$$\epsilon_{n_1} < \frac{1}{2}, \quad \epsilon_{n_2} < \frac{1}{2^2}, \quad \epsilon_{n_3} < \frac{1}{2^3}, \cdots.$$

Let

$$b_n = \frac{1}{n_1}, \quad 0 \leqq n < n_1; \quad b_n = \frac{1}{n_2 - n_1}, \quad n_1 \leqq n < n_2;$$
$$b_n = \frac{1}{n_3 - n_2}, \quad n_2 \leqq n < n_3, \cdots.$$

Then obviously

$$\sum_{n=0}^{\infty} b_n = \infty.$$

But

$$\sum_{n=n_1}^{\infty} b_n \epsilon_n \leqq \sum_{k=2}^{\infty} \sum_{n_{k-1}}^{n_k - 1} \frac{\epsilon_{n_{k-1}}}{n_k - n_{k-1}} \leqq \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = 1.$$

Let

$$h(t) = \frac{b_n}{2^n}, \qquad 2^n \leqq t < 2^{n+1}. \tag{40.18}$$

Then

$$\int_1^{\infty} h(t)\,dt = \sum_{n=0}^{\infty} \frac{b_n}{2^n} \int_{2^n}^{2^{n+1}} dt = \sum_{n=0}^{\infty} b_n = \infty. \tag{40.19}$$

On the other hand,

$$\int_1^{\infty} h(t)\,dt \int_t^{\infty} e^{-\gamma(y)} \frac{dy}{y} \leqq \sum_{n=0}^{\infty} \frac{b_n}{2^n} \int_{2^n}^{2^{n+1}} dt \int_{2^n}^{\infty} e^{-\gamma(y)}\,dy$$
$$= \sum_{n=0}^{\infty} b_n \epsilon_n < \infty.$$

Or integrating by parts

$$\int_1^{\infty} e^{-\gamma(y)} \frac{dy}{y} \int_0^y h(t)\,dt < \infty. \tag{40.20}$$

Let

$$\delta(u) = \log \int_0^u h(t)\,dt.$$

Then by (40.19), $\delta(u) \to \infty$ as $u \to \infty$. By (40.18) $h(t) \leqq 2/t$ since $b_n \leqq 1$. Since

$$\delta'(u) = \frac{h(u)}{\int_0^u h(t)\,dt} < \frac{2}{u \int_0^u h(t)\,dt},$$

relation (40.16) follows from (40.19). Inequality (40.17) follows at once from (40.20).

Lemma 40.4. *If Theorem* LIX *is satisfied and* $f(x) = O(1)$, *then*

$$a_n = O(\lambda_n^{c_{10}}). \tag{40.21}$$

Proof of Lemma 40.3. If we show that we can invert the order of summation and integration in

$$\int_{-\infty}^{\infty} R_n(x, B)\, dx \sum_1^{\infty} a_k \int_{\lambda_k - x}^{\infty} \frac{e^{-y^2/2}}{(2\pi)^{1/2}}\, dy = \int_{-\infty}^{\infty} R_n(x, B) f(x)\, dx,$$

then by Lemma 40.2

$$|a_n| \leqq \limsup_{B \to \infty} \int_{-\infty}^{\infty} |R_n(x, B) f(x)|\, dx = O(\lambda_n^{-10}).$$

Thus this lemma depends on showing that the order of summation and integration above can be inverted. This can be done if

$$\sum_1^{\infty} |a_k| \int_{-\infty}^{\infty} |R_n(x, B)|\, dx \int_{\lambda_k - x}^{\infty} e^{-y^2/2}\, dy < \infty. \tag{40.22}$$

Clearly

$$\begin{aligned} J_1(k) &= \int_{-\infty}^{(\log \lambda_k)/L} |R_n(x, B)|\, dx \int_{\lambda_k - x}^{\infty} e^{-y^2/2}\, dy \\ &\leqq \int_{-\infty}^{(\log \lambda_k)/L} |R_n(x, B)|\, dx \int_{\lambda_k - (\log \lambda_k)/L}^{\infty} y e^{-y^2/2}\, dy \\ &\leqq \exp\{-\tfrac{1}{2}\lambda_k^2 + (\lambda_k \log \lambda_k)/L - \tfrac{1}{2}(\log \lambda_k)^2/L^2\} \int_{-\infty}^{\infty} |R_n(x, B)|\, dx. \end{aligned} \tag{40.23}$$

Also

$$\begin{aligned} J_2(k) &= \int_{\lambda_k/2}^{\infty} |R_n(x, B)|\, dx \int_{\lambda_k - x}^{\infty} e^{-y^2/2}\, dy \\ &\leqq \int_{-\infty}^{\infty} e^{-y^2/2}\, dy \int_{\lambda_k/2}^{\infty} |R_n(x, B)|\, dx \leqq 10 \int_{\lambda_k/2}^{\infty} |R_n(x, B)|\, dx. \end{aligned} \tag{40.24}$$

Finally

$$\begin{aligned} J_3(k) &= \int_{(\log \lambda_k)/L}^{\lambda_k/2} |R_n(x, B)|\, dx \int_{\lambda_k - x}^{\infty} e^{-y^2/2}\, dy \\ &\leqq \int_{(\log \lambda_k)/L}^{\lambda_k/2} |R_n(x, B)|\, e^{-(\lambda_k - x)^2/2}\, dx \\ &\leqq e^{-\lambda_k^2/2} \max_{(\log \lambda_k)/L \leqq x \leqq \lambda_k/2} |R_n(x, B) e^{\lambda_k x}| \int_{-\infty}^{\infty} e^{-x^2/2}\, dx. \end{aligned} \tag{40.25}$$

It is clear that (40.22) is equivalent to

$$\sum_{k=1}^{\infty} |a_k| \{J_1(k) + J_2(k) + J_3(k)\} < \infty. \tag{40.26}$$

That

$$\sum_{k=1}^{\infty} |a_k| J_1(k) < \infty \tag{40.27}$$

follows easily from (40.23) and (39.03). We shall see that we can assume that $\theta(x) > (\log x)^3$ for large x. Thus (40.09) gives

$$R_n(x, B) = O(\exp \{\tfrac{3}{2}Lx - (\log c + \tfrac{1}{2}Lx)^{\delta}\}), \qquad x \to \infty,$$

or

$$R_n(x, B) = O(e^{-x - L^3x^3/10}), \qquad x \to \infty.$$

Thus for large λ_k, (40.24) becomes

$$J_2(\lambda_k) = O\left(\int_{\lambda_k/2}^{\infty} e^{\,x - L^3x^3/10}\, dx\right) = O(e^{\,L^3\lambda_k^3/100}).$$

This implies that

$$J_2(k) = O(e^{-\lambda_k^2/2}).$$

But this last result and (39.03) give

$$\sum_1^{\infty} |a_k| J_2(k) < \infty.$$

Thus of (40.26) there remains to be proved only

(40.28)
$$\sum_1^{\infty} a_k J_3(k) < \infty.$$

Let

(40.29)
$$\beta(u) = \gamma\left(\frac{Lu}{10M}\right) - \delta\left(\frac{Lu}{10M}\right),$$

where $\delta(u)$ satisfies Lemma 40.3. Then by Lemma 40.3

(40.30)
$$\int_1^{\infty} \frac{e^{\beta(u)}}{u}\, du < \infty.$$

Let

$$g(u) = ue^{\beta(u)}.$$

Then

$$g'(u) = e^{\beta(u)}\{1 + u\beta'(u)\} = e^{\beta(u)}\left\{1 + \frac{Lu}{10M}\left(\gamma'\left(\frac{Lu}{10M}\right) - \delta'\left(\frac{Lu}{10M}\right)\right)\right\}$$

$$\geqq e^{\beta(u)}\left\{1 - \frac{Lu}{10M}\,\delta'\left(\frac{Lu}{10M}\right)\right\}.$$

It follows at once from (40.16) that for large u, $g'(u) > 0$. That is, $g(u)$ is monotone increasing for large u. Let $\theta(u)$ be the inverse function of $g(u)$. Then $\theta(u)$ is monotone increasing for large u. Also by (40.30)

$$\int_1^\infty \frac{du}{g(u)} < \infty.$$

Integrating by parts

$$\int_1^\infty \frac{u}{g^2(u)}\, dg(u) < \infty.$$

Let $x = g(u)$. Then the above inequality gives

$$\int_1^\infty \frac{\theta(x)}{x^2}\, dx < \infty.$$

This is (40.10). Also since $\beta(u) \leqq \gamma(Lu/10M)$, and since $\gamma'(u) < M/u$ implies that $\gamma(u) = O(\log u)$, it follows that $\beta(u) = O(\log u) = O(u^{1/10})$ for large u. Thus for large u

$$g(u) < e^{u^{1/3}}$$

or if $x = g(u)$

$$(\log x)^3 < \theta(x).$$

Thus for large x, $\theta(x)$ satisfies all requirements. Moreover for small x we can change $\theta(x)$ to meet any requirements without affecting anything in the following argument.

By (40.09),

$$\begin{aligned} J_4(k) &= \max_{(\log\lambda_k)/L \leqq x < \infty} |R_n(x, B)e^{\lambda_k x}| \\ &= O\left(\max_{(\log\lambda_k)/L \leqq x < \infty} \exp\left\{\tfrac{3}{2}Lx - \theta(ce^{Lx/2}) + \lambda_k x\right\}\right), \qquad k \to \infty. \end{aligned} \tag{40.31}$$

Let $ce^{Lx/2} = y$. Then

$$J_4(k) = O\left(\max_{c\lambda_k^{1/2} \leqq y < \infty} e^{(3+2\lambda_k/L)(\log y - \log c) - \theta(y)}\right), \qquad k \to \infty.$$

We observe that for large k, $\theta(y)$ is involved only for large y. Let $\theta(y) = u$. Then

$$\begin{aligned} J_4(k) &= O\left(\max_{\theta(c\lambda_k^{1/2}) < u < \infty} e^{(3+2\lambda_k/L)(\log g(u) - \log c) - u}\right), \qquad k \to \infty, \\ &= O\left(\max_{0 \leqq u < \infty} e^{(3+2\lambda_k/L)(\log u + \beta(u) - \log c) - u}\right), \qquad k \to \infty. \end{aligned} \tag{40.32}$$

Let the value of u which makes

$$\left(3 + \frac{2\lambda_k}{L}\right)(\log u + \beta(u)) - u \tag{40.33}$$

a maximum be u_0. Then differentiating,

$$\left(3 + \frac{2\lambda_k}{L}\right)\left(\frac{1}{u_0} + \beta'(u_0)\right) = 1.$$

Or

$$\left(3 + \frac{2\lambda_k}{L}\right)\{1 + u_0\beta'(u_0)\} = u_0.$$

From the definition of $\beta(u)$, (40.29),

$$\beta'(u) < \frac{L}{10M}\gamma'\left(\frac{Lu}{10M}\right).$$

Since $\gamma'(u) < M/u$ this gives

$$\beta'(u) < M/u.$$

Thus

$$\left(3 + \frac{2\lambda_k}{L}\right)(1 + M) \geqq u_0.$$

Or assuming $M > 1$ as we may with no restriction and assuming k large,

$$u_0 < \frac{10M\lambda_k}{L}.$$

Since u_0 makes (40.33) a maximum, (40.32) now gives

$$J_4(k) = O(\exp[(3 + 2\lambda_k/L)\{\log(10M\lambda_k)/L + \beta(10M\lambda_k/L) - \log c\}]).$$

From the definition of $\beta(u)$, this gives with c redefined

$$J_4(k) = O(\exp[(3 + 2\lambda_k/L)\{\log\lambda_k + \gamma(\lambda_k) - \delta(\lambda_k) - \log c\}]).$$

Thus since $\delta(\lambda_k) \to \infty$ as $k \to \infty$,

$$J_4(k) = O(\exp[(2\lambda_k/L)\{\log\lambda_k + \gamma(\lambda_k) - 1\}]), \qquad k \to \infty.$$

Recalling the definition of $J_4(k)$ and using this in (40.25),

$$J_3(k) = O\left(\exp\left[-\tfrac{1}{2}\lambda_k^2 + \frac{2\lambda_k}{L}\log\lambda_k + \frac{2\lambda_k}{L}\gamma(\lambda_k) - \frac{2\lambda_k}{L}\right]\right), \qquad k \to \infty.$$

From this and (39.03) it follows that

$$\sum_1^\infty |a_k| J_3(k) < \infty,$$

and this completes the proof of the lemma.

Proof of Theorem LIX. By (40.21) it follows that

$$a_n = O(e^{\lambda_n}).$$

From this Theorem LIX follows using exactly the same proof as for Theorem LVII.

APPENDIX

41. Theorems used. The following theorems are frequently referred to. They are all contained in Titchmarsh, *Theory of Functions*, Oxford, 1932.

THEOREM A. (Jensen.) *Let $f(z)$ be analytic for $|z| < R$. If $f(0) \neq 0$ and $n(x)$ is the number of zeros of $f(z)$ in $|z| \leqq x$, then for $r < R$*

$$\int_0^r \frac{n(x)}{x}\,dx = \frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})|\,d\theta - \log |f(0)|. \tag{41.01}$$

It follows from this that even if $f(0) = 0$

$$\int_1^r \frac{n(x)}{x}\,dx < \frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})|\,d\theta + A \tag{41.02}$$

for some constant A depending only on $f(z)$.

THEOREM B. (Carleman.) *Let $z = x + iy$ and let $f(z)$ be analytic for $x \geqq 0$. Let the zeros of $f(z)$ in $(1 \leqq |z| \leqq R, |\text{am } z| \leqq \frac{1}{2}\pi)$ be $r_1e^{i\theta_1}, r_2e^{i\theta_2}, \cdots, r_ne^{i\theta_n}$. Then*

$$\begin{aligned} \sum_{k=1}^{n}\left(\frac{1}{r_k} - \frac{r_k}{R^2}\right)\cos\theta_k \leqq \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \log |f(Re^{i\theta})|\cos\theta\,d\theta \\ + \frac{1}{2\pi}\int_1^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right)\log |f(iy)f(-iy)|\,dy + A \end{aligned} \tag{41.03}$$

where A is some constant depending only on $f(z)$.

This theorem can obviously be modified to allow for $f(z)$ analytic for $x > 0$ and continuous for $x \geqq 0$.

A trivial addition to the proof of the above theorem gives

THEOREM B′ *Let $z = x + iy$ and let $f(z)$ be analytic for $x \geqq 0$. Let $f(0) = 1$ and let the zeros in $(|z| \leqq R, |\text{am } z| \leqq \frac{1}{2}\pi)$ be $r_1e^{i\theta_1}, r_2e^{i\theta_2}, \cdots, r_ne^{i\theta_n}$. Then*

$$\begin{aligned} \sum_{k=1}^{n}\left(\frac{1}{r_k} - \frac{r_k}{R^2}\right)\cos\theta_k = \frac{1}{\pi R}\int_{-\pi/2}^{\pi/2} \log |f(Re^{i\theta})|\cos\theta\,d\theta \\ + \frac{1}{2\pi}\int_0^R \left(\frac{1}{y^2} - \frac{1}{R^2}\right)\log |f(iy)f(-iy)|\,dy \\ - \frac{f'(0) + \overline{f'(0)}}{4}. \end{aligned} \tag{41.04}$$

THEOREM C. (Phragmén-Lindelof.) *Let $f(z)$ be analytic in the region between two straight lines making an angle π/α at the origin and on the lines themselves. Suppose that*

(41.05) $$|f(z)| \leqq M$$

on the lines and that as $r \to \infty$

$$f(z) = O(e^{r^\beta}),$$

where $\beta < \alpha$, *uniformly in the angle. Then* (41.05) *holds throughout the region.*

THEOREM C′. (Phragmén-Lindelöf.) *Let* $z = x + iy$ *and let* $f(z)$ *be analytic for* $x \geqq 0$. *Let*

$$f(z) = O(e^{|z|^{3/2}}), \qquad |z| \to \infty,\ |\operatorname{am} z| \leqq \tfrac{1}{2}\pi.$$

If

$$f(x) = O(e^{ax}), \qquad x \to \infty,$$

and if

$$f(iy) = O(|y|^n e^{b|y|}), \qquad |y| \to \infty,$$

then for $(r \to \infty,\ |\theta| \leqq \frac{1}{2}\pi)$,

(41.06) $$f(re^{i\theta}) = O(r^n e^{(a \cos \theta + b|\sin \theta|)r}).$$

THEOREM D. (A special case of the Hadamard factorization theorem.) *If* $f(z)$ *is an entire function such that*

$$f(z) = O(e^{|z|^{3/2}}), \qquad |z| \to \infty,$$

then

$$f(z) = az^n e^{bz} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{z/z_k}$$

where $\{z_k\}$ *are the zeros of* $f(z)$ *not at the origin and* n *is the multiplicity of the zero at the origin.*

THEOREM E. (Fourier transform.) *Let* $f(x) \in L(-\infty, \infty)$. *Let*

$$F(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) e^{iux}\, dx.$$

Then at any point where $f(x)$ *is continuous*

$$f(x) = \lim_{A \to \infty} \frac{1}{(2\pi)^{1/2}} \int_{-A}^{A} F(u) \left(1 - \frac{|u|}{A}\right) e^{-iux}\, du.$$

Also if $f(x) \in L^2(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} |F(u)|^2\, du = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

THEOREM F. (Stirling's formula.)

(41.07) $$\log \Gamma(1 + z) = (z + \tfrac{1}{2}) \log z - z + \tfrac{1}{2} \log 2\pi + o(1)$$

for ($|z| \to \infty$, $|\operatorname{am} z| \leqq \pi - \delta$), ($\delta > 0$), *where each logarithm has its principal value.*

In Chapter IV we require the following theorem on Fourier transforms.[1]

THEOREM G. *Let* $f(x) \in L^p(-\infty, \infty)$, $1 < p \leqq 2$. *Then there exists a function* $F(u) \in L^q(-\infty, \infty)$ *where* $q = p/(p-1)$ *and*

$$\frac{1}{(2\pi)^{1/2}} \int_{-A}^{A} f(x) e^{iux}\, dx$$

converges in the qth mean to $F(u)$ *as* $A \to \infty$. *Also*

$$\left\{ \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} |F(u)|^q\, du \right\}^{1/q} \leqq \left\{ \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} |f(x)|^p\, dx \right\}^{1/p},$$

and wherever $f(x)$ *is continuous*

$$f(x) = \lim_{A \to \infty} \frac{1}{(2\pi)^{1/2}} \int_{-A}^{A} F(u) \left(1 - \frac{|u|}{A}\right) e^{-iux}\, du.$$

In Chapter III we make use of the following result which we shall prove here. Essentially this result bears the same relation to Carleman's theorem as the Poisson-Jensen theorem does to Jensen's theorem.

THEOREM H. *Let* $z = x + iy$ *and let* $f(z)$ *be analytic for* $y \geqq 0$. *Then for* $0 < \theta < \pi$ *and* $0 < r < R$

$$\begin{aligned} \frac{\log |f(re^{i\theta})|}{r \sin \theta} \leqq \frac{1}{\pi} \int_{-R}^{R} \log |f(x)| \Bigg[&\frac{1}{x^2 - 2rx \cos \theta + r^2} \\ &- \frac{R^2}{R^4 - 2rxR^2 \cos \theta + r^2 x^2} \Bigg] dx \\ &+ \frac{2R}{\pi} \int_0^{\pi} \log |f(Re^{i\phi})| \frac{(R^2 - r^2) \sin \phi}{|R^2 e^{2i\phi} - 2rR \cos \theta e^{i\phi} + r^2|^2}\, d\phi. \end{aligned} \tag{41.08}$$

This result will be derived by first proving an equality.

Proof. We consider

$$\frac{1}{2\pi i} \int \log f(\xi) \left[\frac{1}{\xi - re^{i\theta}} - \frac{1}{\xi - re^{-i\theta}} - \frac{r}{r\xi - R^2 e^{i\theta}} + \frac{r}{r\xi - R^2 e^{-i\theta}} \right] d\xi \tag{41.09}$$

taken around the semicircle ($|\xi| = R$, $0 < \operatorname{am} \xi < \pi$) and along $(-R, R)$ in the positive direction and starting from the point $(R, 0)$. We first assume there are no zeros in the semicircle. Taking the real part of the above integral we obtain $r \sin \theta$ multiplied by the right side of (41.08). By the residue theorem, (41.09) is equal to $\log f(re^{i\theta})$ and the real part is $\log |f(re^{i\theta})|$. This proves (41.08) (as an equality) when there are no zeros in the semicircle.

[1] Titchmarsh, *A contribution to the theory of Fourier transforms*, Proceedings of the London Mathematical Society, vol. 23 (1934).

Let there be a zero inside the semicircle at z_1. Let

$$f_1(z) = \frac{z - \bar{z}_1}{z - z_1}\frac{R^2 - \bar{z}_1 z}{R^2 - z_1 z}f(z).$$

Then $f_1(z)$ has no zeros in the semicircle and thus (41.08) holds for $f_1(z)$. But on the semicircle

$$\left|\frac{z - \bar{z}_1}{z - z_1}\frac{R^2 - \bar{z}_1 z}{R^2 - z_1 z}\right| = 1.$$

Thus the right side of (41.08) is the same for $f_1(z)$ as for $f(z)$. The left side becomes

$$\frac{\log |f(re^{i\theta})|}{r \sin \theta} + \frac{1}{r \sin \theta} \log \left|\frac{re^{i\theta} - \bar{z}_1}{re^{i\theta} - z_1}\frac{R^2 - \bar{z}_1 re^{i\theta}}{R^2 - z_1 re^{i\theta}}\right|. \tag{41.10}$$

Since

$$\left|\frac{re^{i\theta} - \bar{z}_1}{re^{i\theta} - z_1}\frac{R^2 - \bar{z}_1 re^{i\theta}}{R^2 - z_1 re^{i\theta}}\right| \geqq 1$$

in the semicircle, the second term in (41.10) is positive and can therefore be dropped giving the inequality (41.08) for $f(z)$. How several zeros would be handled is now obvious.

As regards zeros on the boundary of the semicircle, the path of integration of (41.09) can be deformed so as to exclude these zeros by tracing small semicircles of radius ρ about them and proceeding as above. If we let $\rho \to 0$ then since $\rho \log \rho \to 0$ the integrals on these small semicircles will tend to zero. This completes the proof.

Colloquium Publications

1. H. S. White, *Linear Systems of Curves on Algebraic Surfaces*;
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1905, xii, 187 pp. $3.00
2. E. H. Moore, *Introduction to a Form of General Analysis*;
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E. J. Wilczynski, *Projective Differential Geometry*;
1910, x, 222 pp. Out of print

3_1. G. A. Bliss, *Fundamental Existence Theorems*, 1913; reprinted, 1934, iv, 107 pp. 2.00

3_2. E. Kasner, *Differential-Geometric Aspects of Dynamics*, 1913; reprinted, 1934, iv, 117 pp. 2.00

4. L. E. Dickson, *On Invariants and the Theory of Numbers*;
W. F. Osgood, *Topics in the Theory of Functions of Several Complex Variables*;
1914, xviii, 230 pp. Out of print

5_1. G. C. Evans, *Functionals and Their Applications. Selected Topics, Including Integral Equations*, 1918, xii, 136 pp. Out of print

5_2. O. Veblen, *Analysis Situs*, 1922; second ed., 1931, x, 194 pp. Out of print

6. G. C. Evans, *The Logarithmic Potential. Discontinuous Dirichlet and Neumann Problems*, 1927, viii, 150 pp. 2.00
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